A Behavioral Characterization of Markov Chains and Discrete Time Dynamical Systems using Directed Graphs

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July 30, 2011

Abstract

Using a directed graph, a Markov chain can be treated as a dynamical system over a compact space of bi-infinite sequences, with a flow given by the left shift of a sequence. In this paper, we show that the Morse sets of the finest Morse decomposition on this space can be related to communicating classes of the directed graph by considering lifting the communicating classes to the shift space. Finally, we prove that the flow restricted to these Morse sets is chaotic.

1 Introduction

Much research has gone into both the continuous and discrete time of dynamical systems. Many books, such as [6] consider symbolic dynamical systems as dictated by a finite, complete graph, or as a graph with an irreducible associated transition matrix, such as in [8]. This paper makes generalizations to graphs for which all vertices have in-coming and out-going edges. Initially, focus is given to communicating classes in these graphs for the applications they have to symbolic dynamics. We apply the ideas of Morse sets, chain recurrence, and chaotic sets, concepts previously reserved for continuous time dynamics, to these discrete time symbolic dynamical system, and examines how the three are related in symbolic dynamics.

Recently effort has been made to amalgamate the cases and create hybrid systems, combining the two areas. Generally, however, studies have chosen to asses these systems from one of two perspectives: time or space. Motivation for this paper comes from the desire to create and examine a system that is hybridized in both time and space. To do so it is necessary to understand completely the behavior of the discrete time symbolic dynamical system presented here, and also to have a means of infusing it into a continuous case, which, though it is not presented in this paper, is an area of ongoing research.

The paper begins by introducing key terms used in association with directed graphs. We then introduce a symbolic dynamical system based on a directed graph, and discuss the behavior

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of trajectories within this system. The paper concludes by defining a Morse decomposition for the system. Ideas of chain recurrence and chaos are also discussed.

2 Directed Graphs

In this section, we introduce basic terms and structures relevant to the directed graphs associated with Markov chains that will be used in the remainder of the paper. It is important to have a thorough understanding of these structures and concepts, as they provide the foundation for the rest of the work presented.

Definition 1. A finite directed graph \( G = (V, E) \) is a pair of sets \( V = \{1, ..., n\} \) called vertices, and \( E \subseteq V \times V \), called edges.

Definition 2. Any graph \( G = (V, E) \) has a set \( \mathcal{P} := \{(x_1, ..., x_k), (x_i, x_{i+1}) \in E : i, k \in \mathbb{N}\} \). Any element of \( \mathcal{P} \) is called an admissible path of \( G \).

• Any \( \gamma \in \mathcal{P} \) is said to have length \( k - 1 \).
• For any \( \gamma \in \mathcal{P} \): we define \( \gamma_1 \) and \( \gamma_F \) as the initial and final vertices of \( \gamma \), respectively.
• For any \( \gamma \in \mathcal{P} \) and \( \alpha \in V \): \( \gamma \) is said to contain \( \alpha \) if there is some \( \gamma_i = \alpha \).

The next two definitions give the formal definition of positive and negative orbits for the vertices of a directed graph. These orbits have many useful properties which will become evident shortly in this and later sections.

Definition 3. For any \( \alpha \in V \), the set \( O^+(\alpha) := \{\beta \in V \mid \text{there exists } \gamma \in \mathcal{P} \text{ with } \gamma_1 = \alpha, \gamma_F = \beta\} \).

• For any \( A \subseteq V \), the set \( O^+(A) := \{\beta \in V \mid \text{there exists } \gamma \in \mathcal{P} \text{ with } \gamma_1 \in A, \gamma_F = \beta\} \).

Definition 4. For any \( \alpha \in V \), the set \( O^-(\alpha) := \{\beta \in V \mid \text{there exists } \gamma \in \mathcal{P} \text{ with } \gamma_1 = \beta, \gamma_F = \alpha\} \).

• For any \( A \subseteq V \), the set \( O^-(A) := \{\beta \in V \mid \text{there exists } \gamma \in \mathcal{P} \text{ with } \gamma_1 = \beta, \gamma_F \in A\} \).

Definition 5. The out-degree of any \( \alpha \in V \), denoted \( o(\alpha) \), is the number of \( \gamma \in \mathcal{P} \) with length 1 and \( \gamma_1 = \alpha \).

Definition 6. The in-degree of any \( \alpha \in V \), denoted \( i(\alpha) \), is the number of \( \gamma \in \mathcal{P} \) with length 1 and \( \gamma_F = \alpha \).

Definition 7. Any finite directed graph \( G = (V, E) \) with \( o(\alpha) \geq 1 \); for all \( \alpha \in V \) is called an M-graph.

Definition 8. Any finite directed graph \( G = (V, E) \) with \( o(\alpha), i(\alpha) \geq 1 \); for all \( \alpha \in V \) is called an N-graph.

We now move to the topic of loops in graphs, our definition of which may differ slightly from the definitions of loops presented in other graph-related documents. This discussion will lead to the ideas of variance and invariance and will culminate in the identification of arguably the most important structures involved in this paper: the communicating class.

Definition 9. A loop in \( G = (V, E) \) is any \( \gamma \in \mathcal{P} \) with length \( \geq 1 \) which has \( \gamma_1 = \gamma_F \).

Definition 10. A minimal path between two distinct vertices \( \gamma_1, \gamma_F \in V \) is any \( \gamma \in \mathcal{P} \) with length \( l \) such that there exists no other \( \gamma^* \in \mathcal{P} \) with \( \gamma^*_1 = \gamma_1 \) and \( \gamma^*_f = \gamma_F \) of length \( k < l \).

Lemma 11. A minimal path between distinct vertices \( \gamma_1, \gamma_F \in V \) contains no loops.
Proof. Suppose there exists a minimal path $\gamma \in \mathcal{P}$ with $\gamma_1 \neq \gamma_F$, which contains a loop through $q \in V$. Then there exists $\gamma_i = q$ and $\gamma_j = q$, $i, j \in \mathbb{N}, i \neq j$. Now define $\gamma' = (\gamma_{i+1}, \ldots, \gamma_j)$ and $n, k \in \mathbb{N}$ the lengths of $\gamma$ and $\gamma'$, respectively. Since $\gamma_i = \gamma_j$, $(\gamma_i, \gamma_{j+1}) \in E$. Thus we can construct $\gamma^* = \gamma - \gamma'$ as the path $(\gamma_1, \ldots, \gamma_i, \gamma_{j+1}, \ldots, \gamma_F)$ with $\gamma^*_1 = \gamma_1$ and $\gamma^*_F = \gamma_F$ which goes through $q$ only once. This path has length $n - k < n$, contradicting the minimality of $\gamma$. Therefore a minimal path contains no loops. \(\square\)

Lemma 12. Given any M-graph with $n$ vertices and $\gamma \in \mathcal{P}$ from $\alpha$ to $\beta, \alpha \neq \beta$, there exists a $\gamma' \in \mathcal{P}$ of length at most $n - 1$ such that $\gamma'_1 = \alpha$ and $\gamma'_F = \beta$.

Proof. Assume $\gamma' := (\gamma'_1 = \alpha, \gamma_2, \ldots, \gamma'_{k-1}, \gamma'_k = \beta)$ is a minimal path from $\alpha$ to $\beta$ with length $k$. By Lemma 10, $\gamma$ contains no loops, so there cannot be some $i, j \in \mathbb{N}$ with $i \neq j$ for which $\gamma_i = \gamma_j$. So every element of $V$ appears at most once in $(\gamma_i : i = 1, 2, \ldots)$. Thus $k \leq n$, so the length of $\gamma'$ is at most $n - 1$. \(\square\)

It is trivially true, though still useful to notice that:

Lemma 13. Given any M-graph, $\alpha \in O^+(\alpha)$ if and only if $\alpha \in O^-(\alpha)$.

Proof. If $\alpha \in O^+(\alpha)$, then there exists a $\gamma \in \mathcal{P}$ such that $\gamma_1 = \alpha$ and $\gamma_F = \alpha$. Hence, $\alpha \in O^-(\alpha)$. By a similar argument if $\alpha \in O^-(\alpha)$, then $\alpha \in O^+(\alpha)$. \(\square\)

Lemma 14. Given any M-graph, if $\alpha \in O^+(\alpha)$ or if $\alpha \in O^-(\alpha)$, then there exists a loop through $\alpha$.

Proof. If $\alpha \in O^+(\alpha)$ or $\alpha \in O^-(\alpha)$, there exists $\gamma \in \mathcal{P}$ such that $\gamma_1 = \gamma_F = \alpha$. Therefore, there exists a loop through $\alpha$. \(\square\)

Definition 15. A vertex $\alpha \in V$ is variant if there exists no $\gamma \in \mathcal{P}$ containing $\alpha$ which has $\gamma_1 = \gamma_F$.

Definition 16. A vertex $\beta \in V$ is invariant if every $\gamma \in \mathcal{P}$ containing $\beta$ has $\gamma_F = \beta$.

We now come to discuss the particularly useful sets called communicating classes. These communicating classes will provide important information on the behavior of a Markov chain after successive iterations as well as on the direction of flow within the dynamical system spaces. These are arguably the most useful structures we will deal with in this paper so take note of them as their definition will be the means by which many upcoming statements are proven!

Definition 17. A communicating class in $G$ is a subset $C \subseteq V$ for which two things are true:

1. For all $\alpha, \beta \in C$ there exists $\gamma \in \mathcal{P}$ such that $\gamma_1 = \alpha$ and $\gamma_F = \beta$.
2. There exists no $C' \supset C$ where for all $\alpha', \beta' \in C'$ there exists $\gamma' \in \mathcal{P}$ such that $\gamma'_1 = \alpha'$ and $\gamma'_F = \beta'$. This condition is called maximality.

Remark 18. Communicating classes can be classified further in two ways:

1. A communicating class, $C$, is variant if there exists $\gamma \in \mathcal{P}$ with $\gamma_1 \in C$ and $\gamma_F \notin C$.
2. A communicating class, $C$, is invariant if for all $\gamma \in \mathcal{P}$ with $\gamma_1 \in C, \gamma_F \in C$.

Lemma 19. If $O^+(\alpha) \cap O^-(\alpha) \neq \emptyset$, then $O^+(\alpha) \cap O^-(\alpha)$ is a communicating class.
Proof. 1) Let $\beta, \kappa \in A = O^+(\alpha) \cap O^-(\alpha)$. Then there exists $\gamma$ such that $\gamma_1 = \beta$ and $\gamma_F = \alpha$, and $\gamma'$ such that $\gamma'_1 = \alpha$ and $\gamma'_F = \kappa$. Since $\gamma_F = \gamma'_1$ we can construct $\Gamma = (\gamma_1, \gamma_F, \ldots, \gamma'_F)$ to be an admissible path in $P$ from $\beta$ to $\kappa$. Since $\beta$ and $\kappa$ are arbitrary there exists a forwards and backwards path between any two vertices in $O^+(\alpha) \cap O^-(\alpha)$.

2) Suppose there exists some $A' \subset A$ for which for all $\kappa', \beta' \in A'$ there exists $\gamma' \in P$ such that $\gamma'_1 = \kappa'$ and $\gamma'_F = \beta'$ and $A' \neq O^+(\alpha) \cap O^-(\alpha)$. But since $\alpha \in A'$, for all $\beta' \in A'$ there exists a path in both directions between $\alpha$ and $\beta'$. Then for all $\beta' \in A'$ $\beta' \in O^+(\alpha) \cap O^-(\alpha)$ which implies $A' = O^+(\alpha) \cap O^-(\alpha)$ which is a contradiction. Thus $A$ is maximal. \hfill \Box

Note that the empty set is not a communicating class because the empty set does not meet the requirements for maximality.

Lemma 20. Given $A, B \subset V$ communicating classes in $G$: $A \cap B = \emptyset$.

Proof. Suppose there exists some $\alpha \in V$ such that $\alpha \in A \cap B$. Then by definition of communicating classes $O^+(\alpha) = A \cup B$ and $O^-(\alpha) = A \cup B$. So $O^+(\alpha) \cap O^-(\alpha) = A \cup B$. So by Lemma 19 $A \cup B$ is a communicating class, contradicting the maximality of $A$ and $B$. \hfill \Box

We next define an order on the set of all communicating classes in $G$. This order is not only useful for future lemmas and theorems on directed graphs, but also has strong connections to behavior of the flow on our dynamical systems in the coming sections. Noting the means by which this order is proven to work will also help in understanding the order defined later in the section on dynamical systems.

Theorem 21. For all $\alpha \in V$ : $\alpha$ is in some communicating class if and only if $\alpha$ is in some loop.

Proof. ($\Rightarrow$) Pick $\alpha \in C \subset V$ such that $C$ is a communicating class of $V$. By Definition 17 there exists $\gamma \in P$ such that $\gamma_1 = \alpha$ and $\gamma_F = \alpha$. By Definition 9 this is a loop.

($\Leftarrow$) Suppose $\alpha$ is contained in some loop $\gamma \in P$ of length $k$. Then $\alpha = \gamma_i$ for some $i \in \mathbb{N}$. So we can construct $\gamma' = (\gamma_1, \ldots, \gamma_F, \gamma_2, \ldots, \gamma_i)$ to be an admissible path in $P$ with $\gamma'_1 = \gamma'_F = \alpha$. So $\alpha \in O^+(\alpha)$, and so by Lemma 13 $\alpha \in O^-(\alpha)$. So $\alpha \in O^+(\alpha) \cap O^-(\alpha)$ which by Lemma 19 is a communicating class. \hfill \Box

Thus, the existence of a loop within an M-graph automatically implies the existence of a communicating class. We will usually prove the existence of a loop in order to show existence of a communicating class in the following theorems which will allow us to generalize characteristics from our definition of an M-graph.

Theorem 22. Every M-graph has a path of arbitrary length and contains a communicating class.

Proof. (by induction) Let $G$ be an M-graph. Since $o(\alpha) \geq 1$ for all $\alpha \in V$, any $\alpha \in V$ must have associated at least one $e = (\alpha, \eta) \in E$ which constitutes a path of length 1 in $G$. Assume there exists a path $\gamma \in P$ of length $k$ from $\gamma_1 = \alpha \in V$ to $\gamma_{k+1} = \beta \in V$. Since $o(\gamma_{k+1} = \beta) \geq 1$, there exists a $\alpha' \in V$ with $e' = (\gamma_{k+1}, \alpha') \in E$. Therefore, there exists a path of length $k+1$ from $\gamma_1$ to $\alpha'$. Therefore, $G$ contains some path of arbitrarily large length.

Next, consider an M-graph $G$ with $n$ vertices. From above, we know there exists a path $\gamma$ length $n$. Hence, this path contains $n + 1$ vertices. So for at least 1 pair $1 \leq i < j \leq n + 1$, we have $\gamma_i = \gamma_j$. Therefore, there exists a loop from $\gamma_i$ to $\gamma_j$, which by Theorem 21 implies the existence of a communicating class. \hfill \Box

In the next two lemmas we show that every positive orbit in an M-graph is also an M-graph. These two lemmas will be very useful when we begin to consider point trajectories in our symbolic dynamical system space in the coming sections.
Lemma 23. For all $\alpha \in V$: $O^+(\alpha)$, along with its associated edges

$$E_\alpha = \{(x_i, x_{i+1}) \in E | x_i, x_{i+1} \in O^+(\alpha)\}$$

is an M-graph.

Proof. Consider the graph $(O^+(\alpha), E_\alpha)$. For all $\beta \in O^+(\alpha)$, $\beta$ is an element of the M-graph $G$, so by definition there exists $\eta \in V$ such that $e = (\beta, \eta) \in E$. Since $\beta \in O^+(\alpha)$, for each $\gamma \in \mathcal{P}$ with $\gamma_1 = \beta$ and length $n$, $\gamma_i \in O^+(\alpha)$ for all $i \leq n$. Thus for all $\eta$ such that there exists some $e = (\beta, \eta) \in E$, $\eta \in O^+(\alpha)$, which in addition implies $e \in E_\alpha$. So each element of $O^+(\alpha)$ has out-degree of at least one. So $O^+(\alpha)$ with $E_\alpha$ is an M-graph. \(\square\)

Corollary 24. Any communicating class $C_1$, along with its associated edges

$$E_{C_1} = \{(x_i, x_{i+1}) \in E | x_i, x_{i+1} \in C_1\}$$

is an M-graph.

Proof. Consider a communicating class $C_1$ with $\alpha \in C_1$. Since $\alpha$ is in a communicating class, there exists $\beta \in C_1$, (where $\beta$ may equal $\alpha$) such that $(\alpha, \beta) \in E$. Thus, $O(\alpha) \geq 1$, and thus any communicating class is an M-graph. \(\square\)

Definition 25. Let $S = \{ C \mid C \text{ is a communicating class in } G \}$. We define an order, $\prec$, on $S$ by saying, for $C_i, C_j \in S$, $C_i \prec C_j$ when $C_j \subset O^+(C_i)$.

1. $\prec$ is asymmetric: Suppose there were $C_i, C_j \in S$ such that $C_i \prec C_j$ and $C_j \prec C_i$ and $C_i \neq C_j$. Then $C_i \subset O^+(C_j)$ and $C_i \subset O^+(C_j)$. But then we could let $C_T = C_i \cup C_j$ and then for all $\alpha, \beta \in C_T$ there exists $\gamma \in \mathcal{P}$ such that $\gamma_{11} = \alpha'$ and $\gamma_F = \beta'$ which contradicts the maximality of communicating classes in $G$.

2. $\prec$ is transitive: For communicating classes $C_i, C_j, C_k \subset G$ let $C_i \prec C_j$ and $C_j \prec C_k$. Then $C_j \subset O^+(C_i)$ and $C_k \subset O^+(C_j)$. Hence, there exists a $\alpha \in C_i$ and $\beta \in C_j$ such that there is a path $\gamma_1$ from $\alpha$ to $\beta$. There also exist $\varepsilon \in C_j$ and $\sigma \in C_k$ such that there is a path $\gamma_2$ from $\varepsilon$ to $\sigma$. Since $\beta$ and $\varepsilon$ are both in $C_j$, there exists a path $\gamma_3$ from $\beta$ to $\varepsilon$. Thus, the path $(\gamma_1, \ldots, \gamma_1, \ldots, \gamma_2, \ldots, \gamma_2, \ldots, \gamma_3)$ is a path from $\alpha$ to $\sigma$. Additionally, since there exist paths from any vertex in $C_i$ to $\alpha$ and from any vertex in $C_k$ to $\sigma$, there exists paths from any element of $C_i$ to any element of $C_k$, and thus $C_i \prec C_k$.

Lemma 26. Invariant communicating classes are exactly the maximal elements of $S$.

Proof. $(\Rightarrow)$ Let $C \in S$ be an invariant communicating class of an M-graph $G$ containing an invariant communicating class. Suppose there exists another $C' \in S$, $C \neq C'$, such that $C \prec C'$. Then $C' \subset O^+(C)$, so there exists $\gamma \in \mathcal{P}$ with $\gamma_1 \in C$ and $\gamma_F \in C'$. But by the definition of invariant communicating class, for all $\gamma \in \mathcal{P}$ with $\gamma_1 = \alpha \in C$, $\gamma_F = \beta \in C$. Thus $C$ is not invariant, which is a contradiction.

$(\Leftarrow)$ Choose $C \in S$ such that there exists no $C' \in S$ where $C \prec C'$. Then, since $C$ is a communicating class, $C \subset O^+(C)$. Suppose there exists $\alpha \in O^+(C)$ such that $\alpha \notin C$. By Lemmas 23 and 22, there exists $C' \in S$ where $C' \subset O^+(\alpha)$. But then $C \prec C'$ which is a contradiction. \(\square\)

We will now use this order on the set of all communicating classes in an M-graph to show that along with a general communicating class, every M-graph must also contain an invariant communicating class:

Theorem 27. Every M-graph contains an invariant communicating class.
Proof. Suppose there exists some M-graph, \( G = (E,V) \), containing no invariant communicating class. This would mean that \( S = \{ C \subset V | C \text{ is a communicating class of } G \} \), which we know to be non-empty because every M-graph contains a communicating class, has no maximal elements. So for all \( C_i \in S \), there exists a \( C_j \in S \) such that \( C_i \prec C_j \). Since there are finitely many vertices in \( G \) and since \( \bigcap_{C \in S} C = \{ \emptyset \} \), for all \( C \in S \) there must be some \( C_j \in S \) with \( C_i \subset O^+(C_j) \) and \( C_j \subset O^+(C_i) \). This contradicts the maximality of each communicating class \( C_i \in S \). Thus every M-graph contains an invariant communicating class.

Corollary 28. \((O^+(\alpha),E_\alpha)\) contains at least one invariant communicating class for any vertex \( \alpha \) in an M-graph.

Proof. From Theorem 27, all M-graphs contain at least one invariant communicating class. Since \((O^+(\alpha),E_\alpha)\) is an M-graph by Lemma 23, \((O^+(\alpha),E_\alpha)\) necessarily contains an invariant communicating class.

3 Dynamical Systems on Graphs

Much of what we have previously discussed and proved culminates in the generation and characterization of a symbolic dynamical system from a directed graph. We begin this section with a brief overview devoted to definitions for general dynamical systems. We will then begin to create our space directly from the \( N \)-graphs we became familiar with in the earlier sections. The majority of this section, however, will be focused on the topology, structures and eventual full characterization of this dynamical system.

3.1 On Dynamical Systems

Definition 29. A dynamical system (d.s.) on a metric space \( S \) is given by a map \( \Phi : \mathbb{T} \times S \rightarrow S \) that satisfies \( \Phi(0,x) = x \) and \( \Phi(t+s,x) = \Phi(t,\Phi(s,x)) \) for all \( x \in S \) and all \( t,s \in \mathbb{T} \). \( \Phi \) can be expressed by two different but equivalent notations for \( x, x' \in S \) and \( t \in \mathbb{T} \):

\[
\Phi(t,x) = x' \text{ or } \Phi_t(x) = x'
\]

[2]

Definition 30. A d.s. is 1-sided when \( \mathbb{T} = \mathbb{N} \) or \( \mathbb{T} = \mathbb{R}^+ \).

Definition 31. A d.s. is 2-sided when \( \mathbb{T} = \mathbb{Z} \) or \( \mathbb{T} = \mathbb{R} \).

Lemma 32. Any 2-sided d.s. with mapping \( \Phi_t \) has an inverse mapping \( \Phi_{-t} \) where

\[
\Phi_t \circ \Phi_{-t}(x) = \Phi_{-t} \circ \Phi_t(x) = Id(x) = x
\]

Proof. Since for all \( t,s \in \mathbb{T} \), \( \Phi_t \circ \Phi_s(x) = \Phi_{t+s}(x) \), \( \Phi_t \circ \Phi_{-t}(x) = \Phi_0(x) \). By the definition of a dynamical system, \( \Phi_0(x) = x \). Similarly, \( \Phi_{-t} \circ \Phi_{-t}(x) = \Phi_0(x) = x \). Thus, \( \Phi_{-t} \) is the inverse mapping of \( \Phi_t \).

3.2 Generating Shift Spaces

This next space, \( \Upsilon \), is defined so as to better understand the one we will be working with, \( \Omega \).

Definition 33. The bi-infinite product space \( \Upsilon \) of the set \( V = \{1,...,n\} \) is the set of all bi-infinite sequences \( x = (x_{-1},x_0,x_1,...) \) where \( x_i \in V \) for all \( i \in \mathbb{Z} \).

We now define \( N \)-graphs and begin the generation of our space, \( \Omega \). Note that \( \Omega \) will be equal to \( \Upsilon \) when using a complete graph and will be a strict subset of \( \Upsilon \) otherwise.
Definition 34. A directed graph \( G = (V,E) \) is called an N-graph if for all \( \alpha \in V \), \( o(\alpha), i(\alpha) \geq 1 \).

Definition 35. Given an N-graph \( G = (V,E) \) with \( A \subset V \) and \( \alpha \in V \), we define:

- \( \Omega = \{ ..., x_{-2}, x_{-1}, x_0, x_1, x_2, ... \mid (x_i, x_{i+1}) \in E \} \) to be the shift space of \( G \).
- \( \Omega_A = \{ x \in \Omega \mid x_i \in A \text{ for all } i \in \mathbb{Z} \} \) to be the lift of \( A \).
- \( \Omega_p = \{ x \in \Omega \mid \text{there exists } k \in \mathbb{N} \text{ such that } x_i = x_{i+k} \text{ for all } i \in \mathbb{Z} \} \) to be the space of all periodic sequence points in \( \Omega \).

- For any \( y \in \Omega \), \( D_N^y = \{ x \mid x_j = y_j \forall j \in [-N,N] \} \), to be a cylinder set of order \( N \) which contains points \( x \in \Omega \) that agree with \( y \) in all its entries between \( y_{N+1} \) and \( y_{-N-1} \).
- \( \Omega_\alpha = \{ x \in \Omega \mid x_0 = \alpha \} \) to be the points in \( \Omega \) centered at \( \alpha \).

Lifts are a particularly important part of \( \Omega \) and a thorough understanding of their definitions will be very useful in the following pages.

Definition 36. Given \( x \in \Omega \) and \( i \in \mathbb{Z} \), the mapping \( \pi_i : \Omega \rightarrow V \) is called the projection of \( x \) and is given by \( \pi_i(x) = x_i \in V \).

Projections will serve as a bridge connecting points in \( \Omega \) back to the graph generating \( \Omega \). Clearly they will be used when we wish to apply the properties of an N-graph to \( \Omega \). This next definition is a slight extension, also for this purpose:

Definition 37. For any \( x \in \Omega \), the range of \( x \) is given by \( \text{range}(x) = \bigcup_{i \in \mathbb{Z}} \{ \pi_i(x) \} \).

Now we describe our means of “flow” on \( \Omega \):

Definition 38. We define as the left shift on \( \Omega \) the mapping \( \Phi : \Omega \rightarrow \Omega \) where if \( \Phi(x) = y \), then \( y_i = x_{i+1} \) for all \( i \in \mathbb{Z} \).

Though it is quite simple, we will see the \( \Phi \) will have many interesting consequences for the system, many of which are not entirely intuitive. Next some lemmas on lifts are given, pointing out some of their properties and how they are related to the communicating classes of a graph.

Lemma 39. All non-empty lifts are \( \Phi \)-invariant.

Proof. Let \( A \subset V \) and \( \Omega_A \) be its non-empty lift. Suppose there exists some \( x \in \Omega_A \) such that \( \Phi(x) = x' \notin \Omega_A \). Since \( x' \notin \Omega_A \) then for some \( i \in \mathbb{Z} \), \( \pi_i(x') \notin A \) or for some \( x_i, x_{i+1} \in A \), \( (x_i', x_{i+1}') \notin E \). But if \( \Phi(x) = x' \) then \( \pi_{i-1}(x) = \pi_j x' \notin A \) so \( x \notin \Omega_A \), and if \( (x_i', x_{i+1}') \notin E \), then \( (x_{i-1}, x_i) \notin E \) which means \( x \notin \Omega \), which are both contradictions.

Corollary 40. Because the lifts of communicating classes in an N-graph are non-empty, they are invariant.

Lemma 41. For an N-graph \( G = (V,E) \) and \( \alpha \in V \), \( O^+(\alpha) = \bigcup_{i \in \mathbb{N}} \{ \pi_i(x) \mid x \in \Omega_\alpha \} \)

Proof. (\( \subseteq \)) Choose \( \beta \in O^+(\alpha) \). Then there exists \( \gamma \in \mathbb{P} \) such that \( \gamma_0 = \alpha \) and \( \gamma_i = \beta \) for some \( i \in \mathbb{N} \). Since \( \Omega_\alpha \) is made up of all points in \( \Omega \) which are admissible sequences that have an \( x_0 = \alpha \), there must be some \( x \in \Omega_\alpha \) which has \( x_i = \beta \) for some \( i \in \mathbb{N} \) because \( \gamma \) is an admissible path in \( G \). Thus \( \beta \in \bigcup_{j \in \mathbb{N}} \{ \pi_j(x) \mid x \in \Omega_\alpha \} \).

(\( \supseteq \)) Now pick \( \beta \in \bigcup_{i \in \mathbb{N}} \{ \pi_i(x) \mid x \in \Omega_\alpha \} \). Then there exists some \( x \in \Omega \) which is an admissible sequence and has \( x_0 = \alpha \) and \( x_i = \beta \) for some \( i > 0 \). Thus if there is an admissible forward sequence from \( \alpha \) to \( \beta \), by definition it must be true that there is an admissible path in \( G \) from \( \alpha \) to \( \beta \) so \( \beta \in O^+(\alpha) \).

Lemma 42. Given \( A, B \subset V \) communicating classes of \( G \): \( \Omega_A \cap \Omega_B = \emptyset \).

Proof. Since \( A \) and \( B \) are communicating classes in \( G \), by Lemma 20 they are disjoint. Thus by the definition of lifts, \( \Omega_A \cap \Omega_B = \emptyset \).

We now move on to discussing the topology for \( \Omega \), including a metric and an overview of the properties of some of its subsets.
3.3 Characterizing the Shift Space

In this section a metric is first defined on the space Ω. Once this metric has been shown to be satisfactory the discussion will move more generally to the topology it induces on Ω. Important topics covered will be those of completeness, closed subsets, and compactness, all of which will be immensely useful in the last two subsections.

Definition 43.  
\[ f(x_i, y_i) = \begin{cases} 
0 & \text{when } x_i = y_i \\
1 & \text{when } x_i \neq y_i 
\end{cases} \]

Lemma 44. The function \( d(x, y) = \sum_{i=\infty}^{\infty} \frac{f(x_i, y_i)}{4^{|i|}} \), is an acceptable metric on \( \Upsilon \).

Proof.  
1. (Non-negativity) For any \( x, y \in \Upsilon \) and all \( i \in \mathbb{Z} \), \( \frac{f(x_i, y_i)}{4^{|i|}} \) is a quotient of two nonnegative numbers, so it is nonnegative. Nonnegative reals are closed under addition, so \( d(x, y) \geq 0 \).

2. (Identity of Indiscernibles) Suppose \( x \neq y \). Then there is some \( i \) such that \( x_i \neq y_i \), so \( f(x_i, y_i) = 1 \). Thus, \( d(x, y) = \frac{1}{4^{|i|}} > 0 \). If \( x = y \), then \( x_i = y_i \) for all \( i \), and \( f(x_i, y_i) = 0 \) for all \( i \). So then \( d(x, y) = \sum_{i=\infty}^{\infty} \frac{0}{4^{|i|}} = 0 \). So \( d(x, y) \) is zero if and only if \( x = y \).

3. (Symmetry) Clearly, \( f(a, b) = f(b, a) \), so \( d(x, y) = \sum_{i=\infty}^{\infty} \frac{f(x_i, y_i)}{4^{|i|}} = \sum_{i=\infty}^{\infty} \frac{f(y_i, x_i)}{4^{|i|}} = d(y, x) \).

4. (Triangle Inequality) First, we need to show \( f(k, j) + f(j, l) \geq f(k, l) \). If \( k = l \), the case is trivial. If \( k \neq l \), then \( f(k, l) = 1 \). Consider two cases- \( j = k \) or \( j \neq k \). If \( j = k \), then \( f(k, j) = 1 \). Then \( f(k, j) + f(j, l) = 1 \geq f(k, l) \). If \( j \neq k \), then \( f(k, j) = 1 \), so \( f(k, j) + f(j, l) \geq 1 = f(k, l) \). \( d(x, y) + d(y, z) = \sum_{i=\infty}^{\infty} \frac{f(x_i, y_i) + f(y_i, z_i)}{4^{|i|}} \geq \sum_{i=\infty}^{\infty} \frac{f(x_i, z_i)}{4^{|i|}} = d(x, z) \).

The topology induced by using 4 in the denominator is slightly different from those induced by 2 or 3. In the case of 4, two points in \( \Omega \) that agree in the \( i^{th} \) position, but are different before and beyond, will have a distance less than two points that disagree in the \( i^{th} \) position and agree before and beyond, which is not necessarily the case with 2 or 3. All integers greater than 4 will induce an equivalent metric to that induced by 4, so the choice to use 4 was made for simplicity. We now examine some topological properties of \( \Omega \) and \( \Omega_C \).

Lemma 45. All Cauchy sequences in \( \Omega \) converge to some \( x \) in \( \Upsilon \).

Proof. Let \( \{x^n\}_{n=1}^{\infty} \) be a Cauchy sequence in \( \Omega \). Given \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that \( d(x^n, x^m) < \varepsilon \) for all \( n, m > N \). For all \( i \in \mathbb{Z} \), there exists \( \varepsilon' \) such that \( \frac{1}{4^{|i|}} > \varepsilon' \). Then there exists \( N' \in \mathbb{Z} \) such that \( x^n_i = x^m_i \) for some \( i \in \mathbb{Z} \) for all \( n, m > N' \). Therefore, \( \lim_{n \to \infty} x^n_i \) exists for all \( i \in \mathbb{Z} \), and is in \( V \). Let \( y = (...) \lim_{n \to \infty} x^n_{-1}, \lim_{n \to \infty} x^n_0, \lim_{n \to \infty} x^n_1, \ldots \) \( \in \Upsilon \). Then for all \( \varepsilon > 0 \), there exists \( j \in \mathbb{N} \) such that \( \sum_{i=j}^{\infty} \frac{2}{4^{|i|}} < \varepsilon \). There exists \( N_j, N_{j+1}, \ldots, N_j \) such that \( y_i = x^n_i \) for all \( n > N_i, i \in [-j, j] \). Let \( N = \max\{N_j, \ldots, N_j\} \). So for \( n > N \), \( x^n_i = y_i \), and so \( d(x^n, x) < 2 \sum_{i=j}^{\infty} \frac{2}{4^{|i|}} < \varepsilon \). Thus, \( \{x^n\}_{n=1}^{\infty} \) converges to \( y \).

Lemma 46. The shift operator \( \Phi_t \) is continuous for all \( t \).

Proof. If we show \( \Phi_1 \) is continuous, then \( \Phi_t \) is continuous by the continuity of continuous compositions. Let \( \varepsilon > 0 \) be given. Let \( \delta = \varepsilon/4 \). Pick any \( x, y \in \Omega \) such that
\[
d(x, y) = \sum_{i=\infty}^{\infty} \frac{f(x_i, y_i)}{4^{|i|}} < \delta
\]
\[
d(\Phi_1(x), \Phi_1(y)) = \sum_{-\infty}^{\infty} \frac{f(x_{i+1}, y_{i+1})}{4^{i+1}}
\]

So we can then form:

\[
\frac{1}{4}d(\Phi_1(x), \Phi_1(y)) = \sum_{-\infty}^{\infty} \frac{f(x_{i+1}, y_{i+1})}{4^{i+1}} \leq \sum_{-\infty}^{\infty} \frac{f(x_{i+1}, y_{i+1})}{4^{i+1}}
\]

We then know

\[
\sum_{-\infty}^{N} \frac{1}{4^{|i|}} + \sum_{-\infty}^{-N} \frac{1}{4^{|i|}} = 2 \sum_{N}^{\infty} \frac{1}{4^{|i|}} < \varepsilon.
\]

For any \(y\) in the cylinder set with 2N+1 fixed points about the origin defined by \(D^N = \{x^i | x_j^i = x_j \ \forall j \in [-N, N]\}\), \(d(x, y) \leq 2 \sum_{N}^{\infty} \frac{1}{4^{|i|}} < \varepsilon\). Thus, \(D^N \subset B(x, \varepsilon)\).

This last lemma insures that in any \(\varepsilon\)-ball of point \(p \in \Omega\) it is possible to find other points in the ball which have the same entries as \(p\) from entries \(p_{-N}\) to \(p_N\) for some \(N\). This will be most useful when discussing convergent sequences and especially when it is necessary to construct points with certain properties in \(\varepsilon\)-balls in \(\Omega\). All that will need to be done is form a \(\varepsilon\)-ball and show that it is possible find a point within that ball which is equal to all other points in the ball for at some fixed \(-N\) to \(N\) values but contains a constructed sequence before or beyond those entries.

**Lemma 48.** \(\Omega\) is closed in \(\Upsilon\).

**Proof.** Assume we have a convergent sequence of \(\{x^n\}_{n=1}^{\infty} \in \Omega\) such that \(\lim_{n \to \infty} x^n = y\). In order to prove that \(\Omega\) is closed, we have to show \(y \in \Omega\). Because the sequence is convergent to \(y\), there exists some \(x^n\) such that \(\sum_{N}^{\infty} d(x^n, y) = 0\) for any \(N \in \mathbb{N}\). Assume \(y \notin \Omega\). Then there exists an \(i \in \mathbb{Z}\) such that \((y_i, y_{i+1}) \notin E\). But we know there must exist some \(x^n\) such that \(\sum_{-|i|+1}^{i+1} d(x^n, y) = 0\), so \(x^n\) has \(x_i = y_i\) and \(x_{i+1} = y_{i+1}\). This, by definition of \(\Omega\) implies \(x^n \notin \Omega\), contradicting convergence. Thus, \(y \in \Omega\) and the set is closed.

**Corollary 49.** \(\Omega\) is complete.

**Proof.** \(\Omega\) is closed, and all Cauchy sequences in \(\Omega\) converge, so all Cauchy sequences in \(\Omega\) converge to a point in \(\Omega\).

**Lemma 50.** \(\Omega\) is totally bounded.
Proof. Given \( \varepsilon > 0 \), we need to show that there exist a finite collection of \( \varepsilon \)-balls covering \( \Omega \). Take \( N \) such that
\[
\sum_{i=-N}^{-1} \frac{1}{4^{|i|}} + \sum_{i=N}^{\infty} \frac{1}{4^{|i|}} < \varepsilon.
\]
This ensures that any cylinder set of order \( N \) will be contained in a ball of radius \( \varepsilon \). If the graph generating \( \Omega \) has \( M \) vertices, then there exist at most \( M^{2N+1} \) possible cylinder sets of order \( N \) in \( \varepsilon \). Take a collection of \( M^{2N+1} \) \( \varepsilon \)-balls, each of which covers a cylinder set of order \( N \). Clearly, this collection of \( \varepsilon \)-balls is finite. And, since any element of \( \Omega \) is contained in some cylinder set of order \( N \), the collection of \( \varepsilon \)-balls covers \( \Omega \). Hence, \( \Omega \) is totally bounded.

\[\square\]

**Theorem 51.** \( \Omega \) is compact.

**Proof.** By Corollary 49 and Lemma 50, \( \Omega \) is complete and totally bounded, and thus is compact. \[\square\]

**Lemma 52.** For any \( C \subset V \) a communicating class in \( G \): \( \Omega_C \) is closed.

**Proof.** Assume we have a convergent sequence of \( \{x^n\}_{n=1}^{\infty} \in \Omega_C \) such that \( \lim_{n \to \infty} x^n = x \). In order to prove that \( \Omega_C \) is closed, we have to show \( x \in \Omega_C \). Because the sequence is convergent to \( x \), for all \( \varepsilon > 0 \), there exists an \( N \in \mathbb{N} \) such that for all \( n \geq N \), \( d(x_n, x) < \varepsilon \). Assume \( x \notin \Omega_C \). Therefore, there exists an entry of \( x, x_i \), such that \( x_i \notin C \). Hence \( d(x^n, x) \geq \frac{1}{4^{|i|}} \), contradicting convergence. Thus, \( x \in \Omega_C \) and the set is closed.

\[\square\]

**Corollary 53.** For any \( C \subset V \) a communicating class in \( G \): since \( \Omega_C \) is a closed subset of the compact space \( \Omega \), it follows that \( \Omega_C \) is compact.

Finally, we will prove the following lemma on periodic points in \( \Omega \). Its significance will become relevant later when the topic of chaos is examined more closely.

**Lemma 54.** Given \( G \) and associated \( \Omega \): points of \( \Omega_p \) are dense in the lifts of communicating classes.

**Proof.** Pick \( x \in \Omega_C \) and form an \( \varepsilon \)-ball \( B(x, \varepsilon) \) about \( x \) for \( \varepsilon > 0 \). By Lemma 47, \( B(x, \varepsilon) \) contains a cylinder set of \( 2N+1 \) fixed elements about the origin for some \( N \in \mathbb{N} \). Therefore, a periodic sequence of period \( 2N+1+k \) exists such that the repeated portion is the finite subsequence \( (x_{-N}, \ldots, x_0, \ldots, x_N, \ldots, x_{N+k}) \) where \( x_{N+k} \in V \) and \( (x_{N+k}, x_{-N}) \in E \). A periodic sequence of this form can always be found because \( x_{-N} \) and \( x_N \) are in a communicating class together, thus there does exist a path from \( x_N \) to \( x_{-N} \). Hence, because for all \( \varepsilon > 0 \), \( B(x, \varepsilon) \) contains an element of \( \Omega_p \), points of \( \Omega_p \) are dense in \( \Omega_C \).

\[\square\]

Now that an acceptable metric and induced topology on \( \Omega \) have been defined and to some degree described it will be useful to take a more thorough look at how \( \Phi \) affects points in \( \Omega \). To do so we introduce \( \alpha \) and \( \omega \) limit sets, sets common to the study of dynamical systems. These sets will allow us to answer the general question of: what goes where?

### 3.4 \( \alpha \) and \( \omega \) Limit Sets

In this section we define \( \alpha \) and \( \omega \) limit sets for points and subsets of \( \Omega \) and examine some of the consequences of these limit sets. These sets essentially show all of the spaces a given \( x \) could end up after being acted upon by \( \Phi \) in either positive or negative time. Of course, it may often be quite tedious to find a single point in a limit set, but by using our understanding of \( N \)-graphs we will be able to characterize entire sets. In addition it will quickly become evident that these will be particularly useful for a later characterization of behavior on and off of the lifts of communicating classes. For now, we begin with the general definition of \( \alpha \) and \( \omega \) limit sets:
Definition 55. For all \( x \in \Omega \): \( \omega(x) = \{ y \in \Omega \mid \) there exists some sequence \( \{ t_n \}_{n=1}^{\infty} \), with \( t_{n+1} > t_n > 0 \) for all \( n \) and \( t_n \to \infty \), such that \( \lim_{n \to \infty} \Phi_{t_n}(x) = y \). \( \omega(x) \) is called the \( \omega \)-limit set of \( x \).

- For all \( M \subset \Omega \): the \( \omega \)-limit set of \( M \) is denoted by \( \omega(M) = \{ y \in \Omega \mid \) there exist sequences \( \{ x_n \}_{n=1}^{\infty} \) and \( \{ t_n \}_{n=1}^{\infty} \), with \( t_n \to \infty \) with \( t_{n+1} > t_n > 0 \) and \( x^n \in M \) for all \( n \), such that \( \lim_{n \to \infty} \Phi_{t_n}(x^n) = y \).

Definition 56. For all \( x \in \Omega \): \( \alpha(x) = \{ y \in \Omega \mid \) there exist some sequence \( \{ t_n \}_{n=1}^{\infty} \), with \( t_{n+1} < t_n < 0 \) for all \( n \) and \( t_n \to -\infty \), such that \( \lim_{n \to -\infty} \Phi_{t_n}(x) = y \). \( \alpha(x) \) is called the \( \alpha \)-limit set of \( x \).

- For all \( M \subset \Omega \): the \( \alpha \)-limit set of \( M \) is denoted by \( \alpha(M) = \{ y \in \Omega \mid \) there exists sequences \( \{ x_n \}_{n=1}^{\infty} \) and \( \{ t_n \}_{n=1}^{\infty} \) with \( t_n \to -\infty \), with \( t_{n+1} < t_n < 0 \) and \( x^n \in M \) for all \( n \), such that \( \lim_{n \to -\infty} \Phi_{t_n}(x^n) = y \).

Note that these definitions of the limit sets of sets is not equal to the union of the limit sets of the points contained in the set. This next definition given for cycles within \( \Omega \) is a means of characterizing the action of \( \Phi \) on points between two subset of \( \Omega \).

Definition 57. For \( \Omega_A, \Omega_B \subset \Omega \), a cycle between \( \Omega_A \) and \( \Omega_B \) exists if there exist some \( p, q \in \Omega \) such that \( \omega(p) \subset \Omega_A, \omega(q) \subset \Omega_B \) and \( \alpha(p) \subset \Omega_B, \alpha(q) \subset \Omega_A \).

Cycles become very important when Morse Decompositions for \( \Omega \) are discussed in Section 3.5. Before discussing where these cycles can and cannot exist in \( \Omega \), however, it is important to first understand what it means for a point to converge forwards and backwards to two different lifts:

Lemma 58. Given \( A, B \subset V \), their non-empty lifts \( \Omega_A, \Omega_B \subset \Omega \), there exists some point \( p \in \Omega \): \( \alpha(p) \subset \Omega_A \) such that \( \omega(p) \subset \Omega_B \) if and only if there exists \( \gamma \in \mathcal{P} \) with \( \gamma_1 \in A \) and \( \gamma_F \in B \).

Proof. (\( \Rightarrow \)). Assume \( \alpha(p) \subset \Omega_A \) and \( \omega(p) \subset \Omega_B \). Then there exist \( i, j \in \mathbb{Z} \), with \( i < j \) such that for all \( t < i \), \( \pi_t(p) \in A \) and for all \( t > j \), \( \pi_t(p) \in B \). Since \( p \in \Omega \), \( p \) is an admissible bi-infinite path of \( G \), so the subsequence \( \pi_{p-1}, ..., \pi_{p+1} \) of \( p \) is also an admissible path of \( G \). Thus, if we let \( \gamma = (p_{i-1}, ..., p_{j+1}) \), then \( \gamma_1 \in A \) and \( \gamma_F \in B \).

(\( \Leftarrow \)). Assume there exists \( \gamma \in \mathcal{P} \) with \( \gamma_1 \in A \) and \( \gamma_F \in B \) and let \( \gamma \) have length \( k + 1 \). Since \( \Omega_A \) and \( \Omega_B \) are non-empty, and since all non-empty lifts are \( \Phi \)-invariant we can find some \( x \in \Omega_A \) and \( y \in \Omega_B \) such that \( \pi_0(x) = \gamma_1 \in A \) and \( \pi_0(y) = \gamma_F \in B \). Then it is possible to construct \( p \) (\( \gamma, ..., \gamma_F, ..., \gamma \)) with \( \pi_i(p) = \pi_i(x) \) for all \( i \leq 0 \), \( \pi_{k+i}(p) = \pi_i(y) \) for all \( i \geq 0 \), and \( \pi_{i-1}(p) = \gamma_i \) for all \( 1 \leq i \leq k + 1 \). Then clearly \( \alpha(p) \subset A \), \( \omega(p) \subset B \), and \( p \in \Omega \) since \( p \) is made up entirely of admissible paths in \( \mathcal{P} \).

Using this lemma and the definition of communicating classes we can begin to show where these cycles can occur and where they cannot:

Lemma 59. Given an \( \mathcal{N} \)-graph with at least two communicating classes, there are no cycles between the lifts of the communicating classes.

Proof. Choose two communicating classes \( A, B \subset V \) and construct their lifts as \( \Omega_A \) and \( \Omega_B \). Suppose their exists \( p, q \in \Omega \) such that \( \omega(p) \subset \Omega_A, \omega(q) \subset \Omega_B \) and \( \alpha(p) \subset \Omega_B, \alpha(q) \subset \Omega_A \). Then this would constitute a cycle between \( \Omega_A \) and \( \Omega_B \). Since the point \( p \) is an admissible sequence representing an an admissible path in \( G \), we know there must exist a path from vertices in \( A \) to vertices in \( B \) because entries in the left tail of \( p \) must be contained in \( A \) and entries in the right tail of \( p \) must be contained in \( B \). In addition since the point \( q \) is also an admissible sequence representing an admissible path in \( G \), we know there must also exist a path from vertices in \( A \) to vertices in \( B \). The existence of these paths makes the set \( A \cup B \subset V \) a communicating class in \( V \), which contradicts the maximality of \( A \) and \( B \). Thus there can be no cycle between \( \Omega_A \) and \( \Omega_B \).
exists an $N$ for communicating classes after $x$ must be a finite number of communicating classes visited in $x$ in a communicating class. Again, since there are a finite number of communicating classes, there is some point $x$ in the invariant communicating class containing $z$. Then $X = \sum_{i=1}^{\infty} 1_{[i]} \to 0$. Thus, $x$ will converge to $\Omega C$, and hence $\alpha(x) \in \Omega C$. A similar argument holds for communicating classes after $x_j$, and $\omega(x) \subseteq \Omega C_F$ for some communicating class $C_F$.

The other case is that $x$ contains no variant vertices. Thus, every entry of $x$ must be contained in a communicating class. Again, since there are a finite number of communicating classes, there must be a finite number of communicating classes visited before $x_i$, and a finite number of communicating classes after $x_j$. Additionally, because once a sequence leaves a communicating class it can not be revisited (by definition of a communicating class), an order can be defined on the communicating classes in the sequence before $x_i$, and thus there exists a first communicating class, $C_f$. Thus, there exists $m \in \mathbb{Z}$ such that $x_i \in C_f$ for all $i < m$. As $t \to -\infty$, $m \to \infty$, so $\sum_{i=m}^{\infty} 1_{[i]} \to 0$. Thus, $x$ will converge to $\Omega C_f$, and hence $\alpha(x) \in \Omega C_f$. A similar argument holds for communicating classes after $x_j$, and $\omega(x) \subseteq \Omega C_F$ for some communicating class $C_F$.

This characterization becomes very important in the next section when the topic of Morse Decompositions comes up. For now, however, we won't discuss it any further. Instead we look at yet another way limit sets characterize $\Omega$.

**Theorem 61.** Let $X = \{ x \in \Omega \mid \omega(x) \subseteq \Omega C \text{ where } C \subset V \text{ is some invariant communicating class of } G \}$. Then $X$ is open and dense in $\Omega$.

**Proof.** First, we will show that $X$ is dense in $\Omega$. Consider any $y \in \Omega$, and let $\varepsilon > 0$. Then there exists an $N$ such that $\sum_{i=N}^{\infty} \frac{1}{4^i} < \varepsilon$. There is an invariant communicating class in the positive orbit of every vertex in an $\Omega$-graph by corollary 28, so there exists a path $\gamma = \gamma_1, \ldots, \gamma_k$ with $\gamma_1 = y_N$ and $\gamma_k = z$ where $z$ is an element of some invariant communicating class. Thus, there is some point $x$ such that $x = (\ldots, y_{N-1}, y_N, \gamma_2, \ldots, \gamma_k, \gamma'_1, \ldots)$ where $\gamma'$ is some infinite path in the invariant communicating class containing $z$. Then $d(y, x) \leq \sum_{i=N}^{\infty} \frac{1}{4^i} < \varepsilon$, so $x \in B(y, \varepsilon)$. Also, $x \in X$, so because $y$ and $\varepsilon$ were arbitrary, $X$ is dense in $\Omega$.

Now consider any $x \in X$. There must exist some $N$ such that $x_N \in C$ an invariant communicating class of $G$. Let $\varepsilon = \frac{1}{4^N}$. Then for any $y \in B(x, \varepsilon), y_i = x_i$ for all $i = -N, -N+1, \ldots, N$, so $y_N$ is in $C$. Thus, $y_M \in C$ for all $M > N$, and so $\omega(y) \subset \Omega C$. So $y \in X$. Thus, $B_\varepsilon(x) \subset X$, and so $X$ is open in $\Omega$.

The last definition and the lemma that follows it have importance which will become clearer in the next section. For this section, however, it should suffice to see that, using limit sets, we can characterize the action of the shift, $\Phi$, on the lifts of communicating classes as being topologically transitive.
Definition 62. A shift on a metric space $X$ is topologically transitive if there exists some $x \in X$ such that $\omega(x) = X$. [2]

Lemma 63. $\Phi$ is topologically transitive on the lifts of communicating classes: that is, for each communicating class $C_i$, there exists a point $x^*$ such that $\omega(x^*) = \Omega_{C_i}$.

Proof. Consider a communicating class $C_i$. Since within the communicating class there exist a finite number of paths of length $n$ for all $n \in \mathbb{N}$, and the set of loops of length $n$ is a subset of paths of length $n$, there exist a finite number of loops of length $n$ within the communicating class. Consider all loops in the communicating class of length $n$, and let the loops of length $1$ be denoted by $A_1^1, ..., A_1^k$, and loops of length $n$ be denoted by $A_n^1, ..., A_n^{k_n}$. A sequence $x$ can be constructed such that its center portion is

$$(..., A_n^{k_n}, ..., A_1^1, ..., A_1^2, p_1, A_1^1, p_1, A_1^1, ..., A_1^{n-1}, ..., A_n^{k_n}, ...)$$

where $p_1, p_2 \in \{ p | p $ is an admissible path between loops $\}$ which we know to exist because we are operating in the context of communicating classes.

It can now be shown that this sequence converges to every periodic sequence in $\Omega_{C_i}$. Proving this will in turn show that it converges to every point in $\Omega_{C_i}$ because the periodic points are dense in $\Omega_{C_i}$; so if the sequence gets arbitrarily close to any periodic point in $\Omega_{C_i}$, then sequence can get arbitrarily close to every point in $\Omega_{C_i}$. To show that this sequence converges to every periodic point, we must show that every periodic sequence is a subsequence of $x$. For every periodic sequence of period $m$ in $\Omega_{C_i}$, the repeated portion must be a loop of length $m$ in $\Omega_{C_i}$. This loop of length $m$ must appear infinitely throughout $x$ - twice for each multiple $lm$ of $m, l \in \mathbb{N}$. Thus, every periodic sequence is a subsequence of $x$, and $x$ converges to every periodic sequence in $\Omega_{C_i}$.

Corollary 64. This proof also insures that $\alpha(x^*) = \Omega_{C_i}$.

Because this point occurs in the lifts of communicating classes we can use the fact that paths exist between all vertices in any sequence and a similar technique to that used in Theorem 61 to show an extension of the same concept, namely that points for which $\omega(x) = \Omega_{C}$ are dense in $\Omega_c$.

Lemma 65. Given an $N$-graph, $G$, with communicating class $C$, lift $\Omega_{C}$ and constructed point $x^* \in \Omega_{C}$ for which $\omega(x^*) = \Omega_{C}$, we let $Y = \{ x \in \Omega_{C} | \omega(x) = \Omega_{C} \}$. Points of $Y$ are dense in $\Omega_{C}$.

Proof. Pick any $p \in \Omega_{C}$ and form $B(p, \varepsilon)$. Then we know there exists some $N + 1 \in \mathbb{N}$ such that $2 \sum_{n=1}^{N+1} \frac{1}{4^n} < \varepsilon$, so all points $q \in \Omega_{C}$ for which $\sum_{n=1}^{N} \frac{f(p, q)}{4^n} = 0$ are in $B(p, \varepsilon)$. Then, since $\pi_{n}(p) \in C$ and since for all $i \in \mathbb{Z}$, $\pi_i(x^*) \in C$ from the constructed $x^*$, we know there exists $\gamma \in \mathcal{P}$ of some length $k$ for which $\gamma_{1} = \pi_{N}(q)$ and $\gamma_{F} = \pi_{0}(x^*)$. Then it is possible to form $q^* \in B(p, \varepsilon)$ such that $q_{i}^{*} = p_{i}$ for all $-N \leq i \leq N$, $q_{i}^{*} = x_{i}^{*}$ for all $N + k \leq i$, and $q_{i}^{*} = \gamma_{i}$ for all $N \leq i \leq k$. Then, with this construction $\omega(q^*) = \omega(x^*) = \Omega_{C}$ and $q^* \in B(p, \varepsilon)$.

Corollary 66. Not only are points for which $\omega(x) = \Omega_{C}$ dense in $\Omega_{C}$, but points for which $\alpha(x) = \Omega_{C}$ and for which $\alpha(x), \omega(x) = \Omega_{C}$ are also dense in $\Omega_{C}$.

3.5 Communicating Classes: Chain Recurrence, Morse Decompositions, and Chaos

In the previous sections we discussed the flow of individual points and characteristics of certain types of subsets of our space $\Omega$. In a sense, this all serves the purpose of attaining a global understanding of the nature of our dynamical system and the entirety of its flow. However, doing so requires more than the previously discussed $\omega$ and $\alpha$ limit sets. Thus, we introduce the concept of chain recurrence:
Definition 67. Let $\Phi^t$ be a flow on a metric space $(\Omega, d)$. Given $\varepsilon > 0$, $T > 0$ and $x, y \in \Omega$, an $(\varepsilon, T)$-chain from $x$ to $y$ with respect to $\Phi_t$ and $d$ is a pair of finite sequences $x = x_0, x_1, \ldots, x_{n-1}, x_n = y$ in $\Omega$ and $t_0, \ldots, t_{n-1}$ in $[T, \infty)$, denoted together by $(x_0, \ldots, x_n; t_0, \ldots, t_{n-1})$, such that

$$d(\Phi_{t_i}(x_i), x_{i+1}) < \varepsilon$$

for $i = 0, 1, 2, \ldots, n - 1$.  \[1\]

Definition 68. Let $\Phi_t$ be a flow on a metric space $(\Omega, d)$. The forward chain limit set of $x \in \Omega$ with respect to $\Phi_t$ and $d$ is the set $\Psi^+(x) = \bigcap_{\varepsilon, T > 0} \{ y \in \Omega \mid \text{there exists an } (\varepsilon, T)\text{-chain from } x \text{ to } y \text{ with respect to } \Phi_t \}$. The backward chain limit set of $x \in \Omega$ with respect to $\Phi_t$ and $d$ is the set $\Psi^-(x) = \bigcup_{\varepsilon, T > 0} \{ y \in \Omega \mid \text{there exists an } (\varepsilon, T)\text{-chain from } x \text{ to } y \text{ with respect to } \Phi_t \}$. \[1\]

Definition 69. Let $\Phi_t$ be a flow on a metric space $(\Omega, d)$. Two points $x, y \in \Omega$ are chain equivalent with respect to $\Phi_t$ and $d$ if $y \in \Psi^+(x)$ and $x \in \Psi^+(y)$. \[1\]

Definition 70. Let $\Phi_t$ be a flow on a metric space $(\Omega, d)$. A point $x \in \Omega$ is called chain recurrent with respect to $\Phi_t$ and $d$ if $x$ is chain equivalent to itself. The set of all chain recurrent points of $\Phi_t$ is the chain recurrent set of $\Phi_t$. \[1\]

The chain limit sets and chain recurrent points are two more steps towards understanding the flow of the entire system. An educated guess can be made that if all $\omega$ and $\alpha$ limit sets are in lifts of communicating classes then these lifts must have some significance. In the next theorem we show how all chain recurrent points are contained in those lifts and define more of their properties that will be of importance for the Morse Decomposition.

Theorem 71. $x \in \Omega$ is a chain recurrent point if and only if $x \in \Omega_C$ for some $C \subset V$ a communicating class of $G$.

Proof. ($\Rightarrow$). Let $x \notin \Omega_C$ for any $C$, a communicating class of $G$. Then there exists some combination of $i, j, k \in \mathbb{Z}$ with either $\pi_i(x)$ a variant vertex or $\pi_j(x) \in C_1$ and $\pi_k(x) \in C_2$, communicating classes of $G$, where $C_1 \neq C_2$ and $j < k$. We must show that for some $\varepsilon, T$ combination there does not exist an $(\varepsilon, T)$-chain from $x$ to itself.

For the case where there is some $\pi_i(x) \notin C$ for any communicating class, fix $T > 0$ such that $T > i$ and let $\Phi_T(x) = x'$. Then for all $m \geq i$, $\pi_m(x') \in O^+(\pi_i(x))$ and $x'_{n-T} = x_i$. Then for all $\varepsilon < \frac{1}{3^{m+n/2}}$, $q \in B(x', \varepsilon)$ has $\pi_m(q) \in O^+(\pi_i(x))$ for all $m \geq i$ and $q_{n-T} = x_i$. Then all points, $q' \in \Psi^+(q)$, reachable by sequences of $T$'s and $\varepsilon$-jumps from $q$, will have $\pi_i(q') \in O^+(\pi_i(x))$. This will not allow a point $p \in \Omega$ with $p_i = x_i$ because $\pi_i(x)$ is variant, i.e. there is no loop between vertices of $O^+(\pi_i(x))$ and $\pi_i(x)$. Thus $x \notin \Psi^+(x)$.

For the case where there is some $\pi_j(x) \in C_1$ and $\pi_k(x) \in C_2$ where $C_1 \neq C_2$ pick the initial $T$ such that $k - T = j$ and form $\Phi_T(x) = x'$. Then for all $m \geq j$, $\pi_m(x') \in O^+(\pi_k(x))$, $x'_{n-T} = x_j$, and $x'_{n-T} = x_j = x_k$. So for all $\varepsilon < \frac{1}{3^{m+n/2}}$, $q \in B(x', \varepsilon)$, has $j_q = x_k$ for all $m \geq j$. Then, similar to before, all points $q' \in \Psi^+(q)$, reachable by sequences of $T$'s and $\varepsilon$-jumps from $q$ will have $\pi_j(q') \in O^+(\pi_k(x))$. So, again, no point $p \in \Omega$ with $p_j = x_j$ will be reached because by the definition of communicating class there will be no loops between elements of $C_1$ and $O^+(C_2)$. Thus, as before, $x \notin \Psi^+(x)$.

($\Leftarrow$). Let $a = (a_{-1}, a_0, a_1, \ldots)$ and $c = (c_{-1}, c_0, c_1, \ldots)$ with $a, c \in \Omega_C$. Given $\varepsilon, T > 0$ there must exist some $N$ such that $\sum_{i=N}^{\infty} \frac{a_i}{3^i} < \varepsilon$. Let $M = \max(N, T)$. Then let $a' = \Phi_M(a)$. By the definition of communicating class, there is a path $\gamma \in P$ with $\gamma_1 = a'_{M+1}$ and $\gamma_F = c_{-M-1}$. Let $k$ be the length of $\gamma$. Then we can construct $e \in C$ such that $e = (a_{-1}', a_0', \ldots, a_{M}', \gamma_1', \ldots, \gamma_F, c_{-M}, c_{-M-1}, \ldots)$ with $a_0'$ at the origin to be an admissible sequence. The distance then, between $a$ and $e$ is

$$d(a, e') \leq \sum_{i=M+1}^{\infty} \frac{1}{3^i} < \varepsilon.$$ So we can construct an $\varepsilon$-$T$ chain from $a$ to $e$. Similarly we can construct $e' \in C$ such that $e' = \Phi_M(a_{k+1}) = (a_{-1}', a_0', \ldots, a_{M}', \gamma_1', \ldots, \gamma_F, c_{-M}, c_{-M-1}, \ldots)$ with $c_0$ at the origin to be an admissible sequence. This gives us $d(c, e') \leq \sum_{i=-\infty}^{-M-1} \frac{1}{4^i} < \varepsilon$. So
we can make an $\varepsilon$-$T$ chain from $e$ to $c$, and thus we can make an $\varepsilon - T$ chain from $a$ to $c$. Since $a$ and $c$ are arbitrary, it follows that all points in the lift of a communicating class are chain recurrent.

**Definition 72.** A Morse Decomposition of a flow on a compact metric space is a finite collection $\{\mathcal{M}_i, i = 1, \ldots, n\}$ of nonvoid, pairwise disjoint, and isolated compact invariant sets such that:

- (i) For all $x \in \Omega$ one has $\omega(x), \alpha(x) \subset \bigcup_{i=1}^{n} \mathcal{M}_i$.
- (ii) Suppose there are $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_l$ and $x_1, \ldots, x_l \in \Omega \setminus \bigcup_{i=1}^{n} \mathcal{M}_i$ with $\alpha(x_i) \subset \mathcal{M}_{i-1}$ and $\omega(x_i) \subset \mathcal{M}_i$, for $i = 1, \ldots, l$; $\mathcal{M}_0 \neq \mathcal{M}_l$. The elements of a Morse Decomposition are called Morse Sets.

**Remark 73.** A Morse Decomposition $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$ is called finer than a Morse Decomposition $\{\mathcal{M}_1', \ldots, \mathcal{M}_n'\}$, if for all $j \in \{1, \ldots, n\}$ there is $i \in \{1, \ldots, n\}$ with $\mathcal{M}_i \subset \mathcal{M}_j'$, and the containment is strict for at least one $j$.

Up until this point, the only possible candidates for Morse Sets are the lifts of communicating classes. In the next couple of lemmas, we prove that these lifts are isolated after first examining an order defined using $\omega$- and $\alpha$-limit sets between the lifts of communicating classes.

**Definition 74.** Let $\bar{\Omega}_S = \{\Omega_C | C$ is a communicating class of $A\}$. We define $\prec$ on $\bar{\Omega}_S$ as: for $\Omega_1, \Omega_2 \in \bar{\Omega}_S$, $\Omega_1 \prec \Omega_2$ when there exists $p \notin \Omega_1, \Omega_2$ such that $\omega(p) \subset \Omega_2$ and $\alpha(p) \subset \Omega_1$.

**Lemma 75.** $\prec$ is an order on $\bar{\Omega}_S$.

**Proof.**

- $\prec$ is asymmetric: suppose there were $\Omega_1, \Omega_2 \in \bar{\Omega}_S$ such that $\Omega_1 \prec \Omega_2$ and $\Omega_2 \prec \Omega_1$. Then there exists points $p, q \in \Omega$ such that $\omega(p) \subset \Omega_1, \omega(q) \subset \Omega_2$ and $\alpha(p) \subset \Omega_1, \alpha(q) \subset \Omega_2$. But this constitutes a cycle between $\Omega_1$ and $\Omega_2$, which is a contradiction.
- $\prec$ is transitive: assume we have $\Omega_A, \Omega_B, \Omega_C \in \bar{\Omega}_S$ with $\Omega_A \prec \Omega_B$ and $\Omega_B \prec \Omega_C$. So there exists points $p, q \in \Omega$ such that $\omega(p) \subset \Omega_B, \alpha(p) \subset \Omega_A$ and $\omega(q) \subset \Omega_C, \alpha(q) \subset \Omega_B$. Since $\omega(p) \subset \Omega_B, \alpha(p) \subset \Omega_A$ there exists some $N_1, N_2 \in \mathbb{Z}$ such that $\pi_{N_2-k}(p) \notin B$ and $\pi_{N_1-k}(p) \notin A$ for all $k \in \mathbb{N} \cup \{0\}$. Similarly since $\omega(q) \subset \Omega_C, \alpha(q) \subset \Omega_B$ there exists some $N_3, N_4 \in \mathbb{Z}$ such that $\pi_{N_4-k}(p) \notin C$ and $\pi_{N_3-k}(p) \notin A$ for all $k \in \mathbb{N} \cup \{0\}$. Now construct the point $p^*$ to have $\pi_{-k-i}(p^*) = \pi_{N_3-i}$ for all $i \in \mathbb{Z}^-$, and $p_{k+i}(p^*) = \pi_{N_4+i}$ for all $i \in \mathbb{Z}^+$. We leave the values $\pi_i(p^*)$ for $-k \leq i \leq k$ to be filled in by the minimal path between $\pi_{N_2}(p) \in B$ and $\pi_{N_3}(q) \in B$ which we know exists because they are in the same communicating class. So from our construction we can see that $\alpha(p^*) \subset \Omega_A$ and $\omega(p^*) \subset \Omega_C$. Thus $\Omega_A \prec \Omega_C$.

**Definition 76.** An invariant subset $R \subset \Omega$ is isolated if given a neighborhood $N$ of $R$, for all $x \in \Omega$ if $\omega(x) \subset N$ and $\alpha(x) \subset N$, then $x \in R$. [2]

**Lemma 77.** Given a communicating class $C \subset V$ and its lift, $\Omega_C$: $\Omega_C$ is isolated.

**Proof.** Let $B$ be a neighborhood of $\Omega_C$, and let $x \in \Omega$ such that $\omega(x), \alpha(x) \subset B$. For any $p \in B$ there exists some $q \in \Omega_C$ such that $d(p, q) < \varepsilon$. Since $\alpha(x), \omega(x) \subset B$, there exist $i, j \in \mathbb{Z}$ such that for all $t \leq i$ and $t \geq j$ one has $x \in C$ and $\omega(x) \subset C$. Now consider any $k$ such that $i < k < j$. Since $x$ is an admissible sequence there exists $y \in \mathcal{P}$ with $y_1 = \pi_i(x), y_f = \pi_f(x)$ and $y_m = \pi_k(x)$ where $y = (y_1, \ldots, y_m, \ldots, y_f)$. Since $\pi_i(x), \pi_j(x) \in C$ there exists $y' \in \mathcal{P}$ with $y'_1 = \pi_i(x)$ and $y'_f = \pi_j(x)$. So we can construct paths $(y_m, \ldots, y_f), (y_m, \ldots, y'_1, \ldots, y'_f), (y'_1, \ldots, y'_f, y_m, \ldots, y_f) \in \mathcal{P}$ to be between $\pi_i(x), \pi_j(x), \pi_k(x)$. So $\pi_k(x) \in C$. Since $k$ was arbitrary $x \in \Omega_C$.  

\[ \square \]
The next lemma is used to show that the lifts of communicating classes can not be divided into subsets that serve as a finer Morse Decomposition.

**Lemma 78.** Given nonempty and invariant sets $U_1, U_2 \subset \Omega_C$, where $U_1 \cap U_2 = \emptyset$ and $C$ is any communicating class in the $N$-graph $G$, there exists a cycle between $U_1, U_2$.

**Proof.** For any $x \in U_1$ and $\varepsilon > 0$ there exist $B(x, \varepsilon)$ which, by Lemma 47, contains the set \( \{ y \in \Omega \mid q_i = x_i \text{ for } -N \leq i \leq N \text{ and some } N \in \mathbb{N} \} \). Therefore we can construct a point \( r \in B(x, \varepsilon) \) such that \( r_i = x_i \) for all \( i \leq -N \) and \( r_i = z_i \) with \( z \in U_2 \), for all \( i \geq N + k \) where \( k \in \mathbb{N} \) is the length of the minimal path between \( \pi_N(x) \) and \( \pi_{k+1}(z) \). We know this minimal path exists because for all \( i \in \mathbb{Z} \), \( \pi_i(x), \pi_i(z) \in C \) a communicating class. Then since sequences of shifts of \( r \) with \( -t \) will converge to shifts of \( x \in U_1 \), \( \alpha(r) \subset U_1 \) because \( U_1 \) is invariant. Similarly \( \omega(r) \subset U_2 \) because \( U_2 \) is invariant.

With a similar argument a \( y \) can be found in the neighborhood of \( U_2 \) such that \( \alpha(y) \subset U_2 \) and \( \omega(y) \subset U_1 \). Therefore a cycle exist between the two subsets of \( \Omega_C \).

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**Theorem 79.** Given an $N$-graph, $G$, and associated shift space $\Omega \subset \Upsilon$, the lifts of the communicating classes in $G$, represented as elements of $\Omega_S$, are Morse Sets in $\Omega$ which form a finest Morse Decomposition for $\Omega$.

**Proof.** Clearly the set of communicating classes of $G$ is finite because $G$ is finite, so $\Omega_S$ has finitely many elements. Also, by the definition of communicating classes, elements of $\Omega_S$ are nonvoid. Then, from Lemmas 42, 77, 53, and 39 that they are also disjoint, isolated, compact, and invariant. In addition they fulfill the requirements of (i) and (ii) by Lemmas 60 and 59 respectively. Thus the lifts of communicating classes of an $N$-graph, $G$, form a Morse Decomposition for $\Omega$ associated with $G$.

To show that $\Omega_S$ is the finest Morse Decomposition we suppose there exists some finer Morse Decomposition $\mathcal{M} = \{ M_1, \ldots, M_n \}$. Then by definition for all $j \in \{ 1, \ldots, n \}$, where $n$ is the number of communicating classes in $G$, there is $i \in \{ 1, \ldots, n' \}$ with $M_i \subset \Omega_{C_j}$. If $n = n'$, then for each $\Omega_{C_j}$ there exists exactly one $M_i$ such that $M_i \subset \Omega_{C_j}$. If $M_i = \Omega_{C_j}$ for all $i,j$ then $\Omega_S = \mathcal{M}$, so assume there are some $i,j$ where the strict subset applies. Then pick $x' \in \Omega_{C_j}$ such that $x' \notin M_i$. By Lemma 63 there exists $x' \in \Omega_{C_j}$ such that for all $x \in \Omega_{C_j}$, $x \in \omega(x')$. Since $x' \notin M_i$ and $x' \in \omega(x')$, $\omega(x') \not\subset M_i$, and because the lifts of communicating classes are disjoint $\omega(x') \not\subset \bigcup_{i=1}^{n'} M_i$. Thus $\mathcal{M}$ is not a Morse Decomposition because it does not follow requirement (i). If $n < n'$ then for some $\Omega_{C_j}$ there exist $M_i, M_k \subset \Omega_{C_j}$. Since $M_i, M_k$ must be invariant to be Morse Sets, from Lemma 78 we know that there are cycles between $M_i, M_k$, so $\mathcal{M}$ does not follow requirement (ii). Thus $\mathcal{M}$ is not a Morse Decomposition. Therefore, because there exists no finer Morse Decomposition, $\Omega_S$ is the finest Morse Decomposition.

Some $N$-graphs and finite stochastic Markov chain systems contain more than one communicating class. However, in many cases, $\Omega_C$ may be a big part of the space $\Omega$ or its totality, and therefore understand the nature of the flow inside $\Omega_C$ can be of importance. In our system, we will see that chaos will govern the flow within lifts of communicating classes.

**Definition 80.** A flow on a metric space $X$ has sensitive dependence on initial conditions (is chaotic) if there is $\delta > 0$ such that for every $x \in X$ and every neighborhood $B$ of $x$ there are $y \in B$ and $t > 0$ such that $d(\Phi_t(y), \Phi_t(x)) > \delta$. [2]

**Proposition 81.** Consider a flow on a space $X$ that is not a single periodic orbit. If the space is topologically transitive and has a dense subset of periodic points, then it has sensitive dependence on initial conditions.

**Proof.** A proof of this proposition is given in [2].

**Theorem 82.** For an $N$-graph, $G$, and associated shift space $\Omega$: The flow, $\Phi$, has sensitive dependence on initial conditions on the lifts of communicating classes of $G$. 

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Proof. From Lemma 63, \( \Phi \) is topologically transitive on the lifts of communicating classes in \( G \), and from Lemma 54 periodic points are dense in the lifts of communicating classes. Thus, by Proposition 81 the lifts of communicating classes in \( G \) have sensitive dependence on initial conditions.

\[ \square \]

4 Conclusion

We have found that using an \( N \)-graph to generate a symbolic dynamical system implies that the associated shift space will be intricately related to the communicating classes of the graph. In particular, the lifts of these communicating classes in the shift space will define a finest Morse Decomposition for the system. Furthermore, the flow on these lifts are found to have sensitive dependence on initial conditions. All of these characterizations follow essentially from the definitions of communicating class. Future studies may involve examining how these discrete time systems can be ‘injected’ and interact with continuous time dynamical systems in the form of hybrid systems.

5 Acknowledgments

We would like to thank Professor Wolfgang Kliemann, Tracy Mckay, and Geoff Tims, all of Iowa State University, for their assistance with this project. We would also like to thank Iowa State University for their hospitality during this project. In addition, we’d like to thank Alliance and the National Science Foundation for their support of this research.

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