Dissecting Fire Sales Externalities

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Abstract

This paper analyzes the sources of the externalities generated by fire sales in the simplest Walrasian borrower-lender problem. I identify three different channels and I brand them as the risk sharing, collateral and margin channels. The risk sharing channel may create over- or underinvestment; the collateral and margin channels always create overinvestment. Unless transfers are allowed, there are no feasible constrained Pareto improvements. When transfers are allowed, constrained Pareto improvements are generally feasible. This paper recognizes that government interventions are time inconsistent in this environment and shows how the welfare implications of fire sales must be decoupled from amplification mechanisms. Moreover, this paper clearly identifies the effects caused by exogenous and endogenous market incompleteness and its interactions.

JEL numbers: D62, E44, G18.

Keywords: fire sales, pecuniary externalities, collateral constraints, amplification mechanisms, risk sharing, margins, time inconsistency, financial regulation.

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1 Introduction

Fire sales of financial or real assets are recurring phenomena in situations of financial distress. We expect a fire sale to arise when a shock hits the natural owners of a certain asset simultaneously, forcing them to sell their positions to low valuation users less able to hold the asset\(^1\). This description of fire sales, which can be traced back to the seminal paper by Shleifer and Vishny (1992), leaves the following question unresolved: if fire sales are merely due to changes in prices, why should we care about their real effects? Aren’t changes in price a simple redistribution of resources between parties? If we believe that markets are complete, any observed change in prices is merely a visible expression of Walrasian adjustment, which induces transfers between agents that yield a Pareto optimal allocation and make fire sales irrelevant from a welfare perspective. This is the content of the First Welfare Theorem applied to a fire sale context. Therefore, only by carefully understanding the sources and implications of market incompleteness will we be able to determine the real effects generated by fire sales. This is the goal of this paper.

This paper analyzes the different mechanisms by which pecuniary externalities induced by fire sales have first order welfare implications. Using the simplest possible Walrasian model, I am able to isolate and distinguish among three different channels; I brand them as the risk sharing, collateral and margin channels. The main results of my analysis, which contrast with parts of the previous literature, are:

- The risk sharing channel may create over- or underinvestment while the collateral and margin channels always create overinvestment.

- When ex-ante lump sum transfers between agents are not allowed, there are no feasible constrained Pareto improving policies. If ex-ante lump sum transfers are allowed, when credit constraints bind, there are feasible constrained Pareto improving policies. A corollary of these results is that setting a simple capital requirement is not enough in general to achieve a Pareto\(^2\) improvement: ex-ante compensation is required.

- Amplification mechanisms are not the source of the pecuniary externalities described in this paper, even though they can be present in this environment and be significant from a quantitative perspective. Normative and positive implications of fire sales must be decoupled and evaluated separately. I show how we can construct constrained inefficient equilibria without any amplification and efficient equilibria with amplification mechanisms. This sharp distinction is not present in previous literature.

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\(^1\)Shleifer and Vishny (2011) survey the existing literature on fire sales.

\(^2\)This apparently negative result does not necessarily imply that capital regulation is undesirable. It points out, nonetheless, that the policymaker must take into account distributional considerations.
• A planner that seeks to implement constrained Pareto improving policies faces a time inconsistency problem. All three channels are subject to this time inconsistency concern. To my knowledge, this is the first paper that characterizes the time inconsistency problem of the constrained planner in a fire sale environment.

• The amount of fire sold asset is the relevant variable to determine the magnitude of the risk sharing externality; however, the full amount of collateralizable asset held, sold in the fire sale or not, is the pertinent variable to determine the importance of both margin and collateral channels.

• Endogenous market incompleteness creates identical externalities to those implied by exogenous market incompleteness. Endogenous market incompleteness may also be the source of additional externalities that depend on the particular specification of the credit constraints.

The risk sharing channel is due to the interaction between the Walrasian role of prices and the fact that markets are exogenously incomplete. In this paragraph, exogenous means that the underlying financial friction or the particular missing market that originates the incompleteness need not be explicitly determined in the economic environment. The fact that markets are incomplete and that intertemporal marginal rates of substitution across agents are not equalized state by state makes it possible for the planner to induce welfare improving price driven redistribution by changing allocations. Intuitively, a planner can modify allocations to induce price changes that redistribute wealth at each period/state towards those agents with higher marginal utility of wealth. The risk sharing channel on its own can be understood as a particular case of the generic result shown by Geanakoplos and Polemarchakis (1986) and Dreze et al. (1990), in which exogenously incomplete markets are generically constrained inefficient. I want to make clear that only intertemporal risk sharing plays a role in this paper: even when there is no uncertainty, the risk sharing channel is fully active.

Both the collateral and margin channels are created by endogenous market incompleteness; by endogenous I mean that financial constraints are explicitly microfounded and price dependent. The collateral channel arises when price taking agents do not internalize that their decisions can directly modify prices, reducing the borrowing capacity of others. The margin channel works on top of the

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A properly microfounded model in which prices do not affect credit constraints does not generate any externality of this kind. The key distinction for welfare for the collateral and margin channels is whether prices enter explicitly in the credit constraints or not.

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3 Using asset pricing intuition may be helpful here. The Fundamental Theorem of Asset Pricing (see for instance Duffie (2001) or Cochrane (2005)) tells us that, when markets are complete, there exists a unique stochastic discount factor (SDF) or, equivalently, the marginal rate of substitution across any two states is equal for all agents. When markets are incomplete, many SDF’s price assets, allowing different agents to have different marginal rates of substitution.

4 Diamond (1967) and Hart (1975) are the seminal papers in this literature.

5 A properly microfounded model in which prices do not affect credit constraints does not generate any externality of this kind. The key distinction for welfare for the collateral and margin channels is whether prices enter explicitly in the credit constraints or not.
collateral channel if borrowing conditions (margins) react endogenously to prices, as is the case in any model driven by value-at-risk constraints or in situations in which informational asymmetries react to asset prices. The collateral channel can be motivated along the lines of Hart and Moore (1994), where, because of commitment problems, the market value of collateral at the time of repayment arises as the natural borrowing constraint. The margin channel has the spirit of Stiglitz and Weiss (1981), in which prices act as a screening device. It relates more generally to models where asymmetric information plays a key role determining borrowing constraints. Greenwald and Stiglitz (1986) derive general results regarding constrained Pareto inefficiency in these environments. The basic intuition for why these last two channels affect welfare is deceptively simple: when investment opportunities are directly dependent on prices, leaving aside their effects on the budget constraints, price taking behavior need not be conducive to Pareto optimal allocations, since each agent does not internalize that his decisions directly affect the choice sets of other agents in the market. From this viewpoint, prices have failed their full Walrasian duty of signaling scarcity through budget constraints, and begin to play a role as drivers of market imperfections. It shouldn’t be surprising then that price movements induce real external effects. Inefficiency of the competitive equilibrium with endogenous borrowing constraints in Walrasian environments is also discussed by Kehoe and Levine (1993) and Kilenthong and Townsend (2010).

Even though both exogenous and endogenous market incompleteness have been studied before, it is not clear in the literature how they relate to each other. This paper is the first to combine both approaches and to tractably isolate how both frictions affect welfare in the fire sale context. I carry out my analysis in the simplest Walrasian borrower-lender problem, the building block for any model in which financial considerations play a relevant role. For instance, my results apply directly to financial intermediation and housing market problems. A critical reading of the most recent literature that tries to apply these concepts to financial environments suggests that many authors refer very loosely to fire sales and pecuniary externalities as sources of market imperfections, without describing their precise nature. I hope that the holistic framework provided by this paper forces future policy discussions to be explicit about which particular channels are being analyzed.

All my normative results use the notion of constrained Pareto efficiency. The relevant policy question in imperfect economies is not whether a planner can avoid the imperfections and achieve an unconstrained first best outcome, but whether, while being subject to the same constraints as the agents in the decentralized market, a planner can engineer a Pareto improvement. My analysis shows that, in general, when a planner is not allowed to use ex-ante transfers, there are no possible Pareto improvements in a fire sale: the classic Walrasian mechanism, which enters in my model through

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6 The reader may wonder why the generic constrained inefficiency results shown by Geanakoplos and Polemarchakis (1986), Dreze et al. (1990) and Greenwald and Stiglitz (1986) do not apply here unless I allow for ex-ante transfers. The crucial issue is that, in my formulation, the fire sale occurs in a single state, and prices in other states are fixed. With more states, it would be possible to modify allocations in such a way that changes in prices in different states compensate.
the risk sharing channel, is still powerful when fire sales are understood as a single event. Still without using transfers, these negative results can be overturned if borrowers and lenders resources are merged, as happens in representative agent models. Stein (2010) or Woodford (2011) are recent papers that rely on the representative agent assumption, following the Lucas (1990) approach. Those models implicitly allow for an ex-post transfer between buyers and sellers of assets that completely eliminates the risk sharing channel. If ex-ante transfers are allowed, constrained Pareto improvements are feasible.

Depending on the relation between ex-ante and ex-post investment opportunities, the risk sharing channel by itself can generate both over or underinvestment. The collateral channel by itself always induces overinvestment and, as long as margins are countercyclical (the empirically relevant case), the margin channel also pushes towards overinvestment. The effect that ultimately prevails remains an empirical question.

This paper relates to the growing literature on amplification mechanisms and financial constraints in credit markets, shaped by the work of Bernanke and Gertler (1989) and Kiyotaki and Moore (1997). This literature has noticeably been focused on finding mechanisms able to generate enough business cycle amplification, leaving aside the normative implications of financial constraints. Lorenzoni (2008), the closest paper to this one, is an important exception that focuses on the welfare implications of these constraints. His paper shows the possibility of excessive borrowing due to the existence of a pecuniary externality; in my decomposition, the externality highlighted in Lorenzoni’s paper is exclusively of the risk sharing type. My characterization of the externality through an Euler equation greatly improves the exposition of the results.

Even though amplification mechanisms can be relevant for quantitative reasons, I show that feedback loops, cycles or spirals between prices and the amount of assets sold are neither necessary nor sufficient to generate fire sales externalities; in other words, normative and positive implications of fire sales must be decoupled. Previous work by Caballero and Krishnamurthy (2001) and Gromb and Vayanos (2002) also analyzes inefficiencies related to pecuniary externalities in similar environments but with a different focus. Brunnermeier and Sannikov (2010) discuss pecuniary externalities related to the collateral and margin channels. The literature on financial intermediation and markets, for example, Jacklin (1987), Allen and Gale (2004) and Farhi, Golosov and Tsyvinski (2009), is also related to this paper through the risk sharing channel. In that line of work, the possibility of spot retrading in financial markets, together with market incompleteness, reduces risk sharing opportunities, making regulation desirable. Acemoglu and Zilibotti (1997) is an important paper in endogenous market incompleteness and Gale and Gottardi (2010) is a recent related paper focused each other in the right way to induce Pareto improvements. Even though from a purely theoretical perspective my problem could seem irrelevant and nongeneric, if we think that fire sales are situations that happen only in a particular state, then my formulation is the appropriate one.

Krishnamurthy (2010) provides a recent survey of the literature on amplification mechanisms.
on bankruptcy.

In order to understand the actual effects of fire sales externalities, the ultimate goal must be to analyze the magnitudes of these externalities empirically and guide policy intervention. With that goal in mind, and despite the simplicity of my formulation, I have tried to keep this paper as close as possible to a canonical Walrasian framework used in asset pricing and macroeconomics. My hope is that my model will provide a basis to document empirically or evaluate the effect of these externalities in a full fledged quantitative framework. Some recent work by Bianchi and Mendoza (2010), Bianchi (2010) and Jeanne and Korinek (2010), among others, tries to explore the implications of credit constraints quantitatively, but that literature has so far only focused on what I classify as collateral externalities. This paper shows that that line of work has no connection whatsoever with the generic results of Geanakoplos and Polemarchakis (1986) and Dreze et al. (1990). Pulvino (1998) and Benmelech and Bergman (2011) are leading empirical corporate finance papers that also analyze the collateral channel. Consequently, I claim that deeper quantitative work integrating all three mechanisms is still pending.

Section 2 lays out the model and discusses the convenience of the various simplifying assumptions. Section 3 solves the decentralized problem and section 4 analyzes the solution to the planner’s problem, both in the case when ex-ante transfers are allowed and in the case when they are not. Section 5 treats the time inconsistency problem faced by the planner. Section 6 discusses some results and extensions and section 7 concludes. Appendix A contains proofs omitted in the main text and appendix B describes the case with a single noncontingent asset. The online appendix presents additional material.

A reader eager to see the main equation of the paper should go directly to equation (27).

2 The model

The model has two periods $t = 0, 1$ and two states at $t = 1$, denoted as $G$ (good state) and $B$ (bad state). The probability of the good state is $\pi_G$ and the probability of the bad state is $\pi_B = 1 - \pi_G$. There are two groups of agents of unit mass. I refer to them as entrepreneurs and outside consumers, respectively. The entrepreneurs represent the natural holders of capital or high valuation users and are subject to credit constraints. The outside consumers are unconstrained in equilibrium and represent low valuation users or unsophisticated holders of capital. There are two types of goods: consumption and capital goods.

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8 Parts of that literature use the assumption that the borrowing capacity of an asset depends on its current price and not on its future price. That formulation does not rule out default and is hard to rationalize with any microfounded model.

9 A version of the model without uncertainty (i.e., with a single state) is enough to show the most basic results. See the online appendix for that formulation.
2.1 Entrepreneurs

Entrepreneurs are risk averse expected utility maximizers with discount factor 1 (for simplicity), so they maximize $U(C_0) + \pi_G U(C_{1G}) + \pi_B U(C_{1B})$. $U(\cdot)$ is increasing and concave: $U' > 0, U'' < 0$. $C_0, C_{1G}$ and $C_{1B}$ denote the consumption of the entrepreneurs at $t = 0$ and at the good and bad state at $t = 1$ respectively. Entrepreneurs have endowments of the consumption good, $e_0, e_{1G}$ and $e_{1B}$. These agents own a technology that transforms capital at $t = 0$ into $A_G$ or $A_B$ units of the consumption good at $t = 1$ depending on the state. $k_0$ and $p_0$ respectively denote the amount of capital chosen by the entrepreneurs and its market price at $t = 0$. The consumption good can be transformed into capital on a one-to-one basis\(^\text{10}\) and vice versa at $t = 0$, which implies that $p_0 = 1$. $p_{1G}$ and $p_{1B}$ denote the price of capital in each state at $t = 1$. Since $t = 1$ is the last period, the capital held from $t = 0$ to $t = 1$ becomes useless for the entrepreneurs, so they are forced to sell all their capital at $t = 1$ to the outside consumers in the market. The outside consumers fully absorb the capital from the entrepreneurs, generating a fire sale; they are glad to receive it because they are able to transform capital into consumption goods in the same period, right before the world ends.

Entrepreneurs can trade at $t = 0$ in two state contingent assets, one for each state. These assets have $t = 0$ prices $q_{0G}$ and $q_{0B}$ and deliver one unit of the consumption good at each respective state. $a_{0G}$ and $a_{0B}$ denote the net position in each asset, i.e., $a_{0s} > 0$ means that the entrepreneurs are holding the state $s$ asset (saving) and $a_{0s} < 0$ means that they are selling it (borrowing). Outside consumers, described below, take the opposite side of these trades. The time subscript for each endogenous variables, 0 or 1, denotes the period in which the variable is determined.

Entrepreneurs face borrowing and lending constraints in the amounts that they can buy or sell of each contingent asset. For each state $s = \{G, B\}$, their asset holdings must satisfy the following constraints:

$$- [1 - \xi(p_{1s})] p_{1s} k_0 \leq a_{0s} \leq L_s \quad (1)$$

The borrowing constraint $- [1 - \xi(p_{1s})] p_{1s} k_0 \leq a_{0s}$ implies that the maximum amount borrowed by the entrepreneurs depends on the value of the capital in the future, $p_{1s} k_0$, corrected by a pledgeability coefficient $1 - \xi(p_{1s})$, which I make explicitly price dependent. We can think of the function $\xi(p_{1s}) \in [0, 1]$ as the margin required when the entrepreneurs borrow against collateral with future value $p_{1s} k_0$. Without the margin correction, i.e., assuming $\xi(p_{1s}) = 0$, $\forall p_{1s}$, this constraint can be robustly microfounded through a commitment problem à la Hart and Moore (1994). The outside consumers are only willing to lend up to the market value of collateral $p_{1s} k_0$, since entrepreneurs cannot commit to keep their promises and the outside consumers can only recover their investment by seizing the capital. This type of borrowing constraint rules out default. An alternative formulation in which a portion of the returns in the consumption good $A_s k_0$ is pledgeable would not modify\(^\text{10}\)This is the assumption made, for instance, in the basic version of the neoclassical growth model.
any of my results. Note that not all of $A_s k_0$ could be pledged, since that would make the markets effectively complete.

A constant margin, $\xi(p_{1s}) = \xi$, $\forall p_{1s}$, can be easily microfounded as a fixed transaction cost that an entrepreneur pays if he defaults. There are different theories that justify the existence of price dependent margins $\xi(p_{1s})$. By appealing to those, I leave the underlying contractual frictions that determine margins open to interpretation. My formulation is meant to incorporate the ideas developed in models with endogenous margins, such as Brunnermeier and Pedersen (2008), Geanakoplos (2009) and Shin (2010), among others. Value-at-risk constraints or information asymmetries, particularly in adverse selection problems, are the natural microfoundations for these endogenous margins. Contagion effects or search related disruptions could also be considered as the source of $\xi(p_{1s})$. I assume that $\xi(0) = 1$, $\xi(1) = 0$, which implies that there are no margins when borrowing in the good state, and that margins are countercyclical, i.e., $\xi'(\cdot) < 0$, as shown empirically by Adrian and Shin (2010). I have opted for including this channel because of its real world importance. A reader who feels that the margin mechanism lacks an appropriate microfoundation can simply ignore this channel, i.e. set $\xi(p_{1s}) = \xi$, $\forall p_{1s}$, and omit all my references to it throughout the rest of the paper. The conceptual distinction between risk sharing/Walrasian externalities and collateral/frictional externalities remains unchanged, as well as the rest of the results. In that case, the credit constraints would read:

$$-\left[1 - \xi\right] p_{1s} k_0 \leq a_{0s} \leq L_s$$

The lending constraint $a_s \leq L_s$, which prevents entrepreneurs from channeling too much wealth towards a particular state, reflects commitment problems on the part of outside consumers. If we take literally the assumption that only the capital good can be seized, we should expect $L_s = 0$. A positive value for $L_s$ can be rationalized if it denotes the state contingent amount of stores of value in the economy. This constraint can be interpreted along the lines of Holmstrom and Tirole (1998) or Holmstrom and Tirole (2011), who emphasize the importance of outside liquidity/stores of value. Therefore, I assume that $L_s \geq 0$ without loss of generality.

Observe that, in a given equilibrium, at most one credit constraint for each state binds. Which constraint actually binds is crucial for the policy implications of the model.

Note that missing markets in this environment can be thought of extremely tight credit constraints: if we assume for a state $s$ that $L_s = 0$ and $\xi(p_{1s}) = 1$, $\forall p_{1s}$, the model represents a situation in which there is no contingent asset for that state. This is a very natural way to combine the insights of the literatures in both endogenous and exogenous incomplete markets.

A graphical representation of the credit constraints can be seen in figure (1). The distance between $L_s$ and $-\left[1 - \xi(p_{1s})\right] p_{1s} k_0$ is a natural measure of market incompleteness in this model.
2.2 Outside consumers

The outside consumers, who are crucial to simplify the analysis and keep it tractable, are risk neutral, have a discount factor of 1 and have large endowments of the consumption good in each period and state. Because they trade frictionlessly in state contingent securities, their prices are pinned down at $q_0s = \pi_s$. In the good state, they run a linear technology that transforms one unit of capital into the consumption good and vice versa, which implies that the price of capital in the good state is pinned down, $p_{1G} = 1$. In the bad state, the outside consumers use an increasing and concave production technology $F\left(k_{1B}^C\right)$, with $F' > 0$ and $F'' < 0$. Thus, the outside consumers solve $\max_{k_{1B}^C} F\left(k_{1B}^C\right) - p_{1B}k_{1B}^C$, implying an equilibrium price $p_{1B} = F'\left(k_{1B}^C\right)$, where $k_{1B}^C$ is the amount of capital that the outside consumers need to hold in equilibrium. Since they absorb all the capital sold by the entrepreneurs in the bad state, in equilibrium $k_{1B}^C = k_0$, so $p_{1B} = F'(k_0)$. I assume that $p_{1B} = F'(\cdot) \leq 1$. The outside consumers simply represent the unsophisticated agents that must hold the asset during a fire sale, providing a downward sloping demand curve.

Figure (2) shows a simplified representation of the problem.

2.3 Remarks about modeling choices

Many of the assumptions made so far may seem somewhat arbitrary and unrealistic; I want to argue nonetheless that they represent the simplest formulation that encompasses the relevant channels in
my analysis:

- **Number of periods**: the use of only two periods has two important implications. First, all the capital accumulated at \( t = 0 \) must be sold at \( t = 1 \), since it has no other use looking forward. Second, it prevents borrowing from the future as a way to reduce a possible fire sale. Both dimensions can easily be reintroduced into the model at the cost of complicating the analysis, so I leave the extension to three periods for the time inconsistency analysis in section 5. A version of the model with three periods must include at least the two following assumptions: a) the entrepreneurs are net sellers of capital in the bad state and b) the amount of resources that can be raised in the fire sale state is limited\(^{11}\) (i.e., some form of slow moving capital; see Duffie (2010) for a state-of-the-art review of the literature). Assumption b) guarantees that the fire sale cannot be avoided by pledging future returns and assumption a) is crucial to have the right distributional effects of fire sales. In a similar formulation, Lorenzoni (2008) and Gai et al. (2008) allow for multiple equilibria in the fire sale state due to amplification mechanisms. I want to remark that all the inefficiencies treated in this paper are marginal inefficiencies and do not rely on Pareto ranked equilibria. My formulation will make clear that those feedback mechanisms, although realistic and capable of generating additional amplification, are not at all necessary to analyze fire sales. The normative and positive implications of fire sales have different sources and this paper shows how they can be decoupled.

- **Number of states**: a simpler model with a single state and no uncertainty is enough to generate most of the results in the paper. I present that simple version in the online appendix. The main drawback of the single state formulation is that, in what I denominate below\(^{12}\) as case 3 in equilibrium, entrepreneurs would have to be net savers in equilibrium, which may be counterfactual. The current formulation allows entrepreneurs to borrow at \( t = 0 \), by having \( g_0G_0G_0 + q_0B_0a_0B < 0 \), while at the same time arranging insurance towards the fire sale state, that is, \( a_0B > 0 \).

- **Risk averse versus risk neutral agents**: my formulation with risk averse entrepreneurs makes the problem naturally exportable to more quantitative environments. An alternative description, with risk neutral entrepreneurs but endogenous hedging motives due to concave production and imperfect financial markets, along the lines of Froot, Scharfstein and Stein (1993) or Krishnamurthy (2003) is a plausible alternative. Under those assumptions, the entrepreneurs would become effectively risk averse. This paper makes clear that, even though marginal utility of wealth/investment opportunities can be explicitly dependent on prices, as in, for instance,

\(^{11}\)For instance, Shleifer and Vishny (1992) use long term debt to induce debt overhang and Lorenzoni (2008) imposes directly a borrowing constraint to satisfy this requirement.

\(^{12}\)That is the situation in which the entrepreneurs want to arrange insurance for the fire sale state. In a one state model, the entrepreneurs would be net savers.
Gromb and Vayanos (2002) or Lorenzoni (2008), that dependence is not the direct source of the pecuniary externalities. The fact that the outside consumers are risk neutral provides a lot of tractability by pinning down the time zero price of the state contingent securities. Relaxing this risk neutrality assumption would involve accounting for risk and insurance considerations, unnecessarily complicating my analysis.

- **Walrasian versus contracting formulation:** this paper is purposely written in a Walrasian setting, since it is the most natural environment in macroeconomics and for quantitative analysis. Unlike Lorenzoni (2008), all trading is carried out in state contingent securities instead of discretionary contracting. Moreover, the Walrasian formulation makes the role of ex-ante transfers, which can be hidden into the terms of contracting problems, explicit. This restricts the space of transfers, since the agents cannot redistribute resources by modifying the terms of trade in the state contingent assets, and it makes the model easier to scale up to more complex environments.

- **Specification of borrowing constraints:** there is a clear asymmetry in the way the constraints in (1) are written, since the amount that the entrepreneurs can borrow depends on the capital they keep, its price, and a margin while, the maximum feasible amount of savings is capped by an exogenous value. The frictions discussed above that support this specification can be rationalized by assuming that entrepreneurs carry out collateralized borrowing against the value of their assets, while outside consumers have a diversified pool of resources that separates their commitment problem from their specific capital holdings. A crucial difference between this paper and the majority of the recent literature is that the borrowing constraint does not depend on the current price of capital $p_0$ but on its future price $p_1$. The alternative assumption that the borrowing constraint depends on the current price of capital is exploited by Bianchi and Mendoza (2010), Bianchi (2010) and Jeanne and Korinek (2010). Despite its convenience in terms of generating amplification, their formulation lacks an explicit microfoundation and opens the door to the possibility of default, which is implicitly neglected in all those papers.

- **Linear versus concave technology:** I assume a linear technology in the good state for the outside consumers simply to shut down any price fluctuation of capital in the good state. This assumption proxies for the fact that markets for capital are deep in good states, and selling assets does not generate any price impact. In any case, as long as credit constraints do not bind in the good state, any kind of demand for capital works without changing any result. The fact that the price for sold capital is downward sloping in the bad state is, however, crucial for my results. The specific formulation with a concave technology is not necessary and it is used here only for simplicity. An alternative but equivalent formulation could assume that a new group of risk averse agents who lacks diversification opportunities is forced to hold these
fire sold assets, demanding a larger risk premium when they are forced to hold more capital.

3 Decentralized problem

I first set up the full entrepreneur’s problem and subsequently analyze the properties of the competitive equilibrium by incorporating the prices derived from the outside consumers’ optimality conditions. It is important to note that in the decentralized problem, each entrepreneur follows price-taking behavior, unlike the planner in the following section. A representative entrepreneur solves:

\[
\max_{k_0, a_{0G}, a_{0B}, C_0, C_{1G}, C_{1B}} U(C_0) + \pi_G U(C_{1G}) + \pi_B U(C_{1B})
\]  

Subject to the following constraints\(^{13}\): the initial budget constraint (3); the state-by-state budget constraints (4) and (5); and the double set of credit constraints for each state contingent security (6) to (9).

\[
C_0 + p_0 k_0 = e_0 - q_{0G} a_{0G} - q_{0B} a_{0B} \quad (\lambda_0) \tag{3}
\]
\[
C_{1G} = e_{1G} + [p_{1G} + A_G] k_0 + a_{0G} \quad (\pi_G \lambda_{1G}) \tag{4}
\]
\[
C_{1B} = e_{1B} + [p_{1B} + A_B] k_0 + a_{0B} \quad (\pi_B \lambda_{1B}) \tag{5}
\]
\[
a_{0G} \leq \mathcal{T}_G \quad (\pi_G \eta_{0G}) \tag{6}
\]
\[
a_{0G} \geq - [1 - \xi(p_{1G})] p_{1G} k_0 \quad (\pi_G \nu_{0G}) \tag{7}
\]
\[
a_{0B} \leq \mathcal{T}_B \quad (\pi_B \eta_{0B}) \tag{8}
\]
\[
a_{0B} \geq - [1 - \xi(p_{1B})] p_{1B} k_0 \quad (\pi_B \nu_{0B}) \tag{9}
\]

The optimality conditions for the entrepreneurs problem are characterized by:

\[
C_0 : U'(C_0) = \lambda_0 \quad C_{1G} : U'(C_{1G}) = \lambda_{1G} \quad C_{1B} : U'(C_{1B}) = \lambda_{1B}
\]  

\[
a_{0G} : q_{0G} \lambda_0 = \pi_G \lambda_{1G} - \pi_G \eta_{0G} + \pi_G \nu_{0G} \tag{11}
\]
\[
a_{0B} : q_{0B} \lambda_0 = \pi_B \lambda_{1B} - \pi_B \eta_{0B} + \pi_B \nu_{0B} \tag{12}
\]
\[
k_0 : p_0 \lambda_0 = \pi_B \lambda_{1B} (p_{1B} + A_B) + \pi_G \lambda_{1G} (p_{1G} + A_G) + \pi_G \nu_{0G} [1 - \xi(p_{1G})] p_{1G} + \pi_B \nu_{0B} [1 - \xi(p_{1B})] p_{1B} \tag{13}
\]

A competitive equilibrium in this context is a set of allocations \(k_0, a_{0G}, a_{0B}, C_0, C_{1B}, C_{1G}\) and prices \(p_0, p_{1G}, p_{1B}, q_{0B}, q_{0G}\) such that both entrepreneurs and outside consumers behave optimally, given prices, and markets clear.

\(^{13}\)The parentheses represent the Lagrange multipliers used for each constraint. The multipliers are defined to be nonnegative.
Given the assumptions made about the behavior of outside consumers and the available
technologies, the equilibrium prices are pinned down to \( p_0 = 1, \, p_1 G = 1, \, q_0 B = \pi_B, \, q_0 G = \pi_G \) and \( p_1 B = F'(k_0) \). The first order conditions with respect to \( a_0 G, \, a_0 B \) and \( k_0 \) can then be written in equilibrium\(^{14}\) as:

\[
\lambda_0 = \lambda_1 G - \eta_0 G + \nu_0 G \quad (14) \\
\lambda_0 = \lambda_1 B - \eta_0 B + \nu_0 B \quad (15)
\]

\[
1 = \mathbb{E} \left[ \frac{\lambda_1 s}{\lambda_0} (p_1 s + A_s) + \frac{\nu_0 s}{\lambda_0} [1 - \xi (p_1 s)] p_1 s \right] \quad (16)
\]

Condition (16) is a standard Euler equation for the capital good. A variational argument allows us to derive it. \( \lambda_0 \) represents the marginal cost at \( t = 0 \) of an additional unit of capital (if \( p_0 \neq 1 \), this would be \( p_0 \lambda_0 \)), and in equilibrium it must be equal to the expected marginal benefit, \( (p_1 s + A_s) \) units of wealth valued at \( \lambda_1 s, \) in addition to the value that the new unit of capital generates when relaxing the borrowing constraint, \( [1 - \xi (p_1 s)] p_1 s \) new units of borrowing capacity valued at \( \nu_0 s. \)

The first term inside the expectation, \( \frac{\lambda_1 s}{\lambda_0} (p_1 s + A_s) \), represents the classical asset pricing condition and the additional one, \( \frac{\nu_0 s}{\lambda_0} [1 - \xi (p_1 s)] p_1 s \), represents the collateral value of capital when relaxing the borrowing constraint. Conditions (14) and (15) are equivalent Euler equations for the state contingent assets.

Equation (16) makes clear that constrained agents in decentralized markets value more assets that relax their borrowing constraints. Note how the dividend generated by the investment, \( A_s, \) does not carry an extra premium because it is not collateralizable. A similar idea has recently been emphasized by Garleanu and Pedersen (2010) in a CAPM environment. Moreover, if we had different traded assets, and there were an ex-ante production stage, those assets with higher collateral value would be created to the detriment of those that do not increase borrowing capacity.

I now proceed to characterize the 3 different cases that can arise in equilibrium depending on which credit constraints bind. Note that the key variables that determine which borrowing constraints actually bind are the marginal utility of wealth in each period/state, \( \lambda_0 \) and \( \lambda_1 s; \) these are endogenous objects that depend on endowments, borrowing capacity, available technologies and asset prices. Remember that, at the margin, any entrepreneur must be indifferent between consuming and investing.

I have purposely assigned a leading role to the Lagrange multipliers \((\lambda, \eta, \nu)\) in the paper, since they make my results widely applicable. If we had chosen a more general model, e.g., with Epstein-\(^{14}\)By assuming a sufficiently well-behaved \( F'(\cdot) \), a unique solution is guaranteed. Even if there were multiple equilibria, all of them must satisfy (14) to (16) and my analysis would be locally valid.

\( ^{14}\)By assuming a sufficiently well-behaved \( F'(\cdot) \), a unique solution is guaranteed. Even if there were multiple equilibria, all of them must satisfy (14) to (16) and my analysis would be locally valid.
Zin preferences or habit formation, we would only need to look at the corresponding values for the marginal utility of wealth to extrapolate the results of this paper.

In order to simplify the analysis, I make the following assumption.

**Assumption 1.** a) In equilibrium, the marginal utility in the bad state is strictly larger than in the good state: \( \lambda_{1B} > \lambda_{1G} \).  
   b) In equilibrium, the marginal utility at time \( t = 0 \) is strictly larger than the marginal utility in the good state: \( \lambda_0 > \lambda_{1G} \). Depending on the rest of parameters, adjusting the endowments\(^{15}\) is enough to satisfy this assumption.

Assumption 1.a) is innocuous and it just makes the bad state worse than the good state. Assumption 1.b) forces the entrepreneur to always hit their borrowing constraints towards the good state and simply reduces the number of uninteresting cases to analyze in equilibrium.

Combining the first order conditions and making use of assumption 1, an equilibrium can be characterized in the following way:

**Proposition 1.** Depending on parameter values, there are three different cases that arise in equilibrium:

- **Case 1:** Entrepreneurs hit the borrowing constraint towards both good and bad states: \( \lambda_0 > \lambda_{1B} > \lambda_{1G} \). Their asset positions satisfy: 
  \[
  a_{0G} = -[1 - \xi (p_{1G})] p_{1G} k_0 \quad \text{and} \quad a_{0B} = -[1 - \xi (p_{1B})] p_{1B} k_0. 
  \]

- **Case 2:** Entrepreneurs hit the borrowing constraint only towards the good state but not the saving limit or the borrowing constraint towards the bad state: \( \lambda_{1B} = \lambda_0 > \lambda_{1G} \). Their asset positions satisfy: 
  \[
  a_{0G} = -[1 - \xi (p_{1G})] p_{1G} k_0 \quad \text{and} \quad a_{0B} \in \left( -[1 - \xi (p_{1B})] p_{1B} k_0, L_B \right). 
  \]

- **Case 3:** Entrepreneurs hit the borrowing constraint towards the good state and the saving limit towards the bad state: \( \lambda_{1B} > \lambda_0 > \lambda_{1G} \). Their asset positions satisfy: 
  \[
  a_{0G} = -[1 - \xi (p_{1G})] p_{1G} k_0 \quad \text{and} \quad a_{0B} = L_B. 
  \]

Controlling for risk aversion and depending on how attractive investment opportunities are at \( t = 0 \), the entrepreneurs follow a pecking order in borrowing: if the marginal utility of wealth is really large ex-ante, they will borrow as much as they can, pledging income in both good and bad states. If investment opportunities are not so attractive ex-ante, i.e., \( \lambda_0 < \lambda_{1B} \), they will start shifting resources towards the bad state, in which marginal utility of wealth is higher. The intuition for two states extends naturally to a model with several states: in that case, we would find cutoffs such that an entrepreneur borrows the maximum possible amount against states with lowest marginal utility, depending on the rest of the parameters of the model, it may be necessary to set a negative endowment to achieve the desired value for the marginal utility of wealth in each state. This can be understood as a habit or a subsistence level.
stays inside his credit constraints for intermediate states and sends all the savings towards those states with the largest marginal utilities.

The relation between $\lambda_0$ and $\lambda_{1B}$ will be crucial later for the welfare implications of fire sales. If entrepreneurs borrow because they really value wealth at $t = 0$, reducing the size of the fire sale and compensating outside consumers ex-ante will not be optimal from a planner’s perspective. Note also that in case 2 entrepreneurs could be either borrowing or saving in equilibrium, but that will not have any welfare implications.

Figure (3) provides graphical\textsuperscript{16} intuition for proposition 1. Depending on which region $\lambda_0$ falls with respect to $\lambda_G$ and $\lambda_B$, the set of binding constraints differs.

<table>
<thead>
<tr>
<th>Case 2</th>
<th>Case 3</th>
<th>Case 1</th>
</tr>
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<tbody>
<tr>
<td>$\nu_0G &gt; 0$</td>
<td>$\nu_0G &gt; 0$</td>
<td>$\nu_0G &gt; 0$</td>
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<td>$\eta_0G = 0$</td>
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<td>$\nu_0B = 0$</td>
<td>$\nu_0B &gt; 0$</td>
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<td>$\eta_0B = 0$</td>
<td>$\eta_0B &gt; 0$</td>
<td>$\eta_0B = 0$</td>
</tr>
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</table>

Figure 3: Three types of equilibrium cases depending the set of binding constraints.

Without assumption 1, three more cases could arise in equilibrium: $\lambda_0 = \lambda_{1B} = \lambda_{1G}$, in which no constraint binds and markets become effectively complete; $\lambda_{1B} > \lambda_0 = \lambda_{1G}$, in which the entrepreneurs hit the saving limit towards the bad state but not the saving limit or the borrowing constraint towards the good state; and $\lambda_{1B} > \lambda_{1B} > \lambda_0$, in which the savings limit is binding towards both good and bad state. I won’t analyze since them here, since they are empirically less relevant.

Before solving the planner’s problem, it is instructive to characterize the \textbf{first best allocation}. When there are no borrowing constraints, an interior equilibrium is characterized by $\lambda_0 = \lambda_{1G} = \lambda_{1B}$ and an Euler equation for capital $1 = E[(p_{1s} + A_s)]$. In that case, the outside consumers fully insure the entrepreneurs, who can guarantee constant consumptions across all dates and states, while the Euler equation prevents arbitrage between risk-free lending and capital investment.

\textsuperscript{16}My graphical representation seems to imply that $\lambda_G$ and $\lambda_B$ are fixed while $\lambda_0$ varies; of course all three multipliers are jointly determined in equilibrium.
4 Planner’s problem

We know that in the presence of credit constraints, the decentralized equilibrium is trivially not Pareto efficient. A reasonable yardstick to evaluate public intervention instead is that of constrained Pareto efficiency, this is, can a planner improve over the decentralized outcome by using the same instruments as the market? Using this approach, I solve two different versions of a constrained planner’s problem: in the first, the planner is only allowed to modify the allocations chosen by the agents, while in the second, the planner can also design ex-ante transfers at $t = 0$ between the two groups of agents. I make the distinction between these two cases because giving the planner the opportunity to redistribute assets ex-ante may be considered by some as giving the planner more resources than the decentralized market. Note that when solving the constrained efficient problem, the planner, unlike the agents in the decentralized market, internalizes the effect that changes in allocations have on prices.

Note that this paper always uses the notion of ex-post constrained Pareto efficiency, that is, a government intervention is only considered to be a Pareto improvement when both entrepreneurs and outside consumers are (weakly) better off. This is the most demanding welfare criterion. Other authors, for example, Hart and Zingales (2011) in a related environment, implicitly use the less restrictive notion of ex-ante constrained Pareto inefficiency. Under that criterion, a Pareto improvement can be reached by maximizing the sum of the utilities of both types of agents. That criterion is justified by arguing that, ex-ante, there is an equal chance for an agent to be an entrepreneur or an outside consumer. That view of the world implicitly introduces ex-ante insurance opportunities between entrepreneurs and outside consumers. Moreover, the ex-ante formulation is implicitly granting Pareto weights to each set of agents; I believe that that task should not be part of the economic analysis and should simply be left to the judgment of the policymaker.

Figure 4 represents this argument graphically.
Figure 4: Ex-post versus ex-ante constrained Pareto improvements.

$U_C$ denotes the utility of the outside consumers and $U_E$ denotes the utility of the entrepreneurs.

4.1 Transfers at $t = 0$ not allowed

In this case, the planner maximizes the utility of the entrepreneurs subject the additional constraint $V_P^C \geq V_M^C$, where $V_P^C$ and $V_M^C$ denote the indirect utility of the outside consumers under the planner’s dictate ($P$) and the decentralized market outcome ($M$) respectively. The utility of the outside consumers, $V_j^C$ for $j = \{M, P\}$, is given by the sum of their large endowments at each state $e^C = e_0^C + e_G^C + e_B^C$ and the expected profits made in the bad state $F(k_0) - F'(k_0) k_0$. The planner acknowledges that profits for outside consumers are zero in the good state and that $k_0$ must be equal to $k_B^C$ in equilibrium. Given the risk neutrality of outside consumers, any trading in contingent assets does not affect their utility. Therefore, the indirect utility for outside consumers $V_j^C(k_0)$ can be written as a function of only $k_0$:

$$V_j^C(k_0) = e^C + \pi_B [F(k_0) - F'(k_0) k_0]$$

I denote by $\theta$ the Lagrange multiplier for the constraint $V_P^C \geq V_M^C$. In an equivalent formulation of the planner’s problem in which a weighted sum of utilities is maximized, $\theta$ is set exogenously and controls the weight assigned to the outside consumers. I make reference to this interpretation of the problem when discussing my results. The planner’s problem can then be stated as:
\[
\max_{k_0, a_0G, a_0B, C_0, C_1G, C_1B} U(C_0) + \pi_B U(C_{1B}) + \pi_G U(C_{1G}) + \theta \left[ e^C + \pi_B (F(k_0) - F'(k_0) k_0) - V_M^C \right]
\]

The planner faces the same set of constraints that the entrepreneurs, with the caveat that the effect of allocation on prices is directly taken into account.

\[
C_0 + k_0 = e_0 - \pi_G a_0G - \pi_B a_0B \quad (\lambda_0) \tag{17}
\]
\[
C_{1G} = e_{1G} + [1 + A_G] k_0 + a_0G \quad (\pi_G \lambda_{1G}) \tag{18}
\]
\[
C_{1B} = e_{1B} + [F'(k_0) + A_B] k_0 + a_0B \quad (\pi_B \lambda_{1B}) \tag{19}
\]
\[
a_{0G} \leq \mathcal{L}_G \quad (\pi_G \eta_{0G}) \tag{20}
\]
\[
a_{0G} \geq - [1 - \xi (1)] k_0 \quad (\pi_G \nu_{0G}) \tag{21}
\]
\[
a_{0B} \leq \mathcal{L}_B \quad (\pi_B \eta_{0B}) \tag{22}
\]
\[
a_{0B} \geq - [1 - \xi (F'(k_0))] F'(k_0) k_0 \quad (\pi_B \nu_{0B}) \tag{23}
\]

The solution to the planner’s problem without transfers is characterized by the following optimality conditions\textsuperscript{17}:

\[
C_0 : U'(C_0) = \lambda_0 \quad C_{1G} : U'(C_{1G}) = \lambda_{1G} \quad C_{1B} : U'(C_{1B}) = \lambda_{1B} \tag{24}
\]
\[
a_{0G} : \lambda_0 = \lambda_{1G} - \eta_{0G} + \nu_{0G} \tag{25}
\]
\[
a_{0B} : \lambda_0 = \lambda_{1B} - \eta_{0B} + \nu_{0B} \tag{26}
\]

\[
1 = E \left[ \frac{\lambda_{1s}}{\lambda_0} (p_{1s} + A_s) + \frac{\nu_{0s}}{\lambda_0} [1 - \xi (p_{1s})] p_{1s} \right] + \frac{1}{\lambda_0} \pi_B \left[ \begin{array}{c}
(\lambda_{1B} - \theta) \\
\text{Risk sharing}
\end{array} \right] + \nu_{0B} \left[ \begin{array}{c}
(1 - \xi) F' + (1 - \xi) \xi F'' \\
\text{Margin} \quad \text{Collateral}
\end{array} \right] F'' (k_0) \tag{27}
\]

Conditions (24), (25) and (26) are identical to the ones arising in the decentralized problem. The terms that differentiate the Euler condition for \( k_0 \) (27) from its decentralized counterpart (13) are the sources of the externalities.

Expression (27) clearly identifies the three channels through which fire sales can matter. As in the decentralized problem (see equation (16)), the first two terms in the right hand side of (27) represent the classical asset pricing present value and the fact that the agents take into consideration the additional value of \( k_0 \) relaxing the borrowing constraint.

\textsuperscript{17}To simplify the notation, I often write \( F', F'', \xi \) and \( \xi' \) for \( F'(k_0), F''(k_0), \xi (F'(k_0)) \) and \( \xi' (F'(k_0)) \).
The externality term in the planner’s problem has three parts, which I call the risk sharing, margin and collateral channels. The **risk sharing** channel comes from the classic Walrasian function of prices; a reduction (increase) in prices generates redistribution from net sellers (buyers) to net buyers (sellers) of an asset. The planner knows that each marginal unit of capital reduces the wealth of the entrepreneurs by $F''k_0$ in the bad state, and that they value this increase in wealth $\lambda_1B$. On the other hand, the outside consumers make a capital gain with this change, and their gain in utility is $\theta F''k_0$. If $\theta$ is an exogenous weight and its value is lower than $\lambda_1B$, the planner opts for a lower amount of capital $k_0$, reducing the size of the fire sale in the bad state. When $\theta$ is endogenous, I show in proposition 2 that it adjusts to make the externality term equal to zero. The **collateral** channel captures the externality that an agent causes to other agents when he doesn’t internalize the fact that an additional unit of capital depresses the debt capacity of all the other agents. The **margin** channel accounts for the fact that agents do not internalize that changes in prices change the debt capacity directly through a tightening of margins.

Which constraints bind in equilibrium and how the three channels interact determine the optimal $k_0$ for the planner. The risk sharing term of the externality $\lambda_1B - \theta$ has a priori an ambiguous sign, but the second part (collateral and margin) $\nu_0B (-\xi'F' + (1 - \xi))$ is unambiguously nonnegative\(^\text{18}\). When the whole term that corresponds to the three channels is negative (positive), the marginal social returns of $k_0$ are smaller than those assessed privately and the planner wants to choose a smaller (larger) amount of capital than in the decentralized equilibrium. Note that $F'' < 0$. Figure (5) shows graphically\(^\text{19}\) how $k_0$ is determined in equilibrium. The upward sloping line represents $\lambda_0$ (the left hand side in equation (27)) and the downward sloping ones are respectively the private and social marginal benefit of an extra unit of $k_0$ (the right hand side in equation (27)).

\(^{18}\)These results follow directly from the assumption made about the determination of margins: $\xi \in [0, 1]$ and $\xi' < 0$. \(^{19}\)This representation holds under modest conditions about $F'$ and $\xi'$. See online appendix for more details about uniqueness.
Figure 5: Overinvestment case. A graphical analysis of the externalities.

The first important policy implication of this paper is captured in proposition 2:

**Proposition 2.** *If ex-ante transfers are not allowed, the planner finds no Pareto improvements.*

**Proof.** See Appendix A. □

The main intuition in the proof is the following: in order to improve the welfare of the outside consumers, the amount of capital chosen by the planner must be larger than the one in the decentralized market. But this imposes an extra constraint in $k_0$ for the entrepreneurs, reducing their welfare unless $k_0$ is the same as in the decentralized market. In general, if there is only one state in which we are concerned about a fire sale, or in the case in which fire sales induce transfers in the same direction between agents, the Walrasian effect of prices, which I call the risk sharing channel, is sufficient to prevent the existence of constrained Pareto improvements.

Because this model assumes that the amount of equity is fixed (the initial endowment), any increase in $k_0$ can be immediately traced to capital requirements or leverage ratios. This proposition implies that only setting capital requirements, without creating ex-ante transfers, is not enough to create Pareto improvements.

Proposition 2 differs from the recent results in Stein (2010), Bianchi (2010), Bianchi and Mendoza (2010) and Woodford (2011), in which a reduction of the amount of capital is enough to induce a Pareto improvement. In those models, a representative agent formulation merges borrowers and lenders to allow for implicit ex-post transfers. In other words, the authors neglect the Walrasian effect of prices and only focus on the collateral channel. The crucial assumption in those models is
that the transfers between borrowers and lenders are welfare neutral, since the marginal utility of both groups is identical. In my formulation, the term $\lambda^C_{1B} - \theta \lambda^U_{1B}$, becomes exactly zero when $\theta = 1$ and $\lambda^C_{1B} = \lambda^U_{1B}$. In that situation, overinvestment in the decentralized market is the only possible outcome. In general, any model that collapses to a representative agent formulation will face the same concern.

An interesting alternative environment is the one in which we set $\theta = 0$ exogenously, i.e., the planner just wants to maximize the utility of the entrepreneurs. In that case, the new solution is always to reduce capital and implement a $k^\text{planner}_0$ less than $k^\text{market}_0$. This provides a clear rationale for a planner that cares more about entrepreneurs than outside consumers to curtail ex-ante investment and set tight capital requirements.

Some may argue that the representative agent framework, which shuts down the risk sharing channel, is the appropriate one through which to study aggregate models. Note that the same Walrasian mechanism that usually equates marginal rates of substitution to marginal rates of transformation is the one that creates the externality here. Arbitrarily removing that implication would be completely at odds with assuming that competitive markets determine the rest of the variables in the model.

### 4.2 Transfers at $t = 0$ allowed

If the planner can redistribute resources at $t = 0$, the solution to the constrained problem varies considerably. In this new setup, the planner is allowed to design a transfer $T_0$ between entrepreneurs and outside consumers at $t = 0$; $T_0 > 0$ implies a positive transfer from entrepreneurs to consumers and $T_0 < 0$ vice versa.

With this new policy instrument, the problem to solve becomes:

$$\max_{k_0, a_0G, a_0B, C_0, C_{1G}, C_{1B}, T_0} U(C_0) + \pi_B U(C_{1B}) + \pi_G U(C_{1G}) + \theta \left[ e^U + T_0 + \pi_B (F(k_0) - F'(k_0) k_0) - V^U_M \right]$$

Subject to the following budget constraint at $t = 0$:

$$C_0 + k_0 + T_0 = e_0 - \pi_G a_0G - \pi_B a_0B (\lambda_0)$$

Identical budget constraints and borrowing constraints to those stated in (18), (19), (20), (21), (22) and (23) also need to hold.

Therefore, the conditions for optimality are identical to those in (24), (25), (26) and (27), with the addition of:

20The concavity of $U$, the existence of two-sided credit constraints and the fact that entrepreneurs' endowments are finite imply that $T_0$ is bounded. In a situation with risk-neutrality on both sides or without borrowing constraints, it may be necessary to introduce constraints in the amount of $T_0$ to have a bounded solution.
This condition pins down the value of $\theta$. Intuitively, the planner equalizes the benefit of shifting one unit of consumption from entrepreneurs to consumers, represented by $\theta$, to the cost in terms of utility for entrepreneurs, $\lambda_0$.

By combining (28) with the Euler equation for capital, we can find an equivalent expression to (27):

$$
1 = E \left[ \frac{\lambda_1}{\lambda_0} (p_1 s + A_s) + \frac{\nu_0}{\lambda_0} [1 - \xi (p_1 s)] p_1 s \right] + \pi_B \left[ \frac{\lambda_1 B}{\lambda_0} - 1 \right] + \nu_0 B \left[ -\xi' F' + (1 - \xi) \right] F'' k_0
$$

The sign of the risk sharing term depends on the difference between the ratio of marginal utilities for entrepreneurs, $\frac{\lambda_1 B}{\lambda_0}$, and outside consumers, equal to 1 because of their risk neutrality. As I discuss with more detail in section 6, in the case with risk averse outside consumers, the risk sharing term would look like $\left( \frac{\lambda_1 B}{\lambda_0} - \frac{\lambda_1 C}{\lambda_0} \right)$.

With a representative agent formulation, marginal utilities are trivially equal and the risk sharing term disappears. In that case, absent any binding price dependent credit constraint, the planner’s problem has the same solution as the decentralized market. If markets are complete, the intertemporal rates of substitution across states are equalized, $\nu_0 B = 0$ and the whole externality term vanishes. This comes from the First Welfare Theorem with complete markets. The sign of the transfer $T_0$ is easily determined after finding $k^\text{planner}_0$; $T_0$ will be positive when $k^\text{market}_0 > k^\text{planner}_0$ and negative otherwise.

**Proposition 3.** When ex-ante transfers are allowed, as long as $\frac{\lambda_1 B}{\lambda_0} \neq 1$ (i.e., constraints bind and markets are not locally complete) and $\xi > 0$, the planner can improve over the decentralized outcome.

**Proof.** See Appendix A. □

The intuition for this proposition is the following. In case 1, entrepreneurs really value having resources at $t = 0$, so they are better off by enjoying more funds ex-ante. Therefore, the planner is tempted to create a positive transfer from outside consumers to entrepreneurs ex-ante and induce a larger fire sale in the bad state to compensate the outside consumers. In other words, entrepreneurs increase their capital $k_0$ at $t = 0$, when they value wealth the most, and then compensate outside consumers with an increased fire sale in the bad state. Nonetheless, inducing a large fire sale in the bad state reduces the debt capacity of the entrepreneurs towards that state, creating a tradeoff
for the planner through the collateral/margin mechanism. In general, the strength of both effects will determine whether over- or underinvestment with respect to the constrained optimum arises in equilibrium. Under the particular assumptions of this paper, because \( v_0 B = \lambda_1 B - \lambda_0 \), the externality term collapses to \( \left( \frac{\lambda_1}{\lambda_0} - 1 \right) [\xi + \xi' F'] \); hence, unless \( |\xi'| \) is large in absolute value, underinvestment would arise. Note also that if margins are set to zero, the risk sharing and the collateral channel exactly compensate, rendering a constrained efficient decentralized outcome; this is a knife-edge case that need not be true in a general version with more periods or different traded assets\(^{21}\).

Finally, a plausible additional constraint on the planner could be \( T_0 \geq 0 \), this is, ex-ante subsidies to entrepreneurs are not allowed. In that case, the Pareto improvements will exist only in case of overinvestment.

In case 2, entrepreneurs are locally unconstrained, the planner cannot improve over the decentralized outcome and classical Pareto efficiency intuition applies.

In case 3, \( v_0 B = 0 \), so only the risk sharing channel is active. The planner optimally reduces the amount of capital chosen ex-ante by entrepreneurs, dampening the effect of the fire sale and making entrepreneurs better off in the bad state, where they value wealth the most. To make this modification acceptable for outside consumers, a positive transfer from entrepreneurs to outside consumers is designed at \( t = 0 \). Note that in this situation, the entrepreneurs’ borrowing constraint is not binding in the bad state: entrepreneurs recognize that they value wealth a lot in the bad state and they actively arrange insurance towards that state.

5 Time inconsistency

Time inconsistency problems are common in macroeconomic policy\(^{22}\) environments. To the best of my knowledge, this paper provides a novel treatment of the time inconsistency problem faced by a planner in an environment with fire sales externalities. I only solve for the constrained efficient allocation, without discussing its explicit implementation in the form of capital requirements or taxes. However, in a simple environment like this, those interventions are straightforward to characterize.

In order to proceed with this new analysis, I must add a third period to the model. Assume that \( t = 0, 1, 2 \), with the new event tree representing the marginal utility of wealth for entrepreneurs given by figure 6.

\(^{21}\)See appendix B for an example.

I assume that in period \( t = 2 \), capital can be transformed again into the consumption good on a one-to-one basis, \( p_2 = 1 \). Now outside consumers must absorb the amount \( k_0 - k_{1B} \) of the capital good in the bad state. I restrict the analysis to the case in which entrepreneurs are net sellers\(^\ast\) of capital in the bad state, i.e., \( k_0 - k_{1B} > 0 \); this assumption implies that the price of capital in the bad state is \( p_{1B} = F'(k_0 - k_{1B}) \leq 1 \), with \( F'(0) = 1 \). To ease the exposition, I eliminate the price dependence of margins by setting \( \xi(p_{1s}) = \bar{\xi} \), \( \forall p_{1s} \) and I directly analyze the solution for the constrained planner. As shown in the previous section, in order to allow for feasible Pareto improvements, the planner must have the opportunity to use an ex-ante transfer \( T_0 \). The rest of the original assumptions still hold and the notation is extended naturally.

Note that this formulation embeds some notion of within period amplification: the more the entrepreneurs engage in fire sales in the bad state by increasing \( k_0 - k_{1B} \), the deeper the fire sale will be, reducing their net worth and making them more willing to sell even more capital in order to keep consumption high. I analyze amplification and its relation to welfare with more detail in section 6.

Under these new assumptions, the problem solved by the constrained planner at \( t = 0 \) is

\[
\max_{a_0G, a_0B, a_{1G}, a_{1B}, C_0, C_{1G}, C_{1B}, C_{2G}, C_{2B}, k_0, k_{1B}, k_{1B}, T_0} U(C_0) + \mathbb{E}[U(C_{1s}) + U(C_{2s})] + \theta \left[ e^C + T_0 + \pi_B (F(k_0 - k_1) - F'(k_0 - k_1)[k_0 - k_1]) - V_M^C \right]
\]

(30)

With budget constraints:

\[^\ast\text{For this to happen in equilibrium, the endowment of the entrepreneurs must be low in this state (perhaps negative) as well as their productivity } A_{1B}. \text{ They must also have small borrowing capacity in that particular state.}\]
The constrained planner optimality conditions are:

\[ C_0 + k_0 + T_0 = e_0 - \pi_G a_{0G} - \pi_B a_{0B} \quad (\lambda_0) \]
\[ C_{1G} + k_{1G} = e_{1G} + [1 + A_{1G}] k_0 + a_{0G} - a_{1G} \quad (\pi_G \lambda_{1G}) \]
\[ C_{1B} + F'(k_0 - k_{1B}) k_{1B} = e_{1B} + [F'(k_0 - k_{1B}) + A_{1B}] k_0 + a_{0B} - a_{1B} \quad (\pi_B \lambda_{1B}) \]
\[ C_{2B} = e_{2B} + [1 + A_{2B}] k_{1B} + a_{1B} \quad (\pi_B \lambda_{2B}) \]
\[ C_{2G} = e_{2G} + [1 + A_{2G}] k_{1G} + a_{1G} \quad (\pi_G \lambda_{2G}) \]

And credit constraints:

\[ a_{0G} \leq T_{0G} \quad (\pi_G \eta_{0G}G) \quad a_{0G} \geq - \left(1 - \bar{\xi}\right) k_0 \quad (\pi_G \nu_{0GG}) \]
\[ a_{0B} \leq T_{0B} \quad (\pi_B \eta_{0B}) \quad a_{0B} \geq - \left(1 - \bar{\xi}\right) F'(k_0 - k_{1B}) k_0 \quad (\pi_B \nu_{0BG}) \]
\[ a_{1G} \leq T_{1G} \quad (\pi_G \eta_{1G}) \quad a_{1G} \geq - \left(1 - \bar{\xi}\right) k_{1G} \quad (\pi_G \nu_{1G}) \]
\[ a_{1B} \leq T_{1B} \quad (\pi_B \eta_{1B}) \quad a_{1B} \geq - \left(1 - \bar{\xi}\right) k_{1B} \quad (\pi_B \nu_{1BG}) \]

The constrained planner optimality conditions are:

\[ C_0 : U'(C_0) = \lambda_0 \quad C_{1s} : U'(C_1) = \lambda_{1s}, \text{ for } s = G, B \quad C_{2s} : U'(C_2) = \lambda_{2s}, \text{ for } s = G, B \]

Decentralized Market

\[ k_0 : \lambda_0 = \mathbb{E} [\lambda_{1s} (p_{1s} + A_{1s})] + \mathbb{E} [\nu_{0s} p_{1s}] + \pi_B \]

Externality

\[ F''(k_0 - k_{1B}) \]

Risk sharing

Relaxation

Collateral

\[ k_{1G} : \lambda_{1G} = \lambda_{2G} (1 + A_{2G}) + \left(1 - \bar{\xi}\right) \nu_{1G} \]

Decentralized Market

\[ k_{1B} : F'(k_0 - k_{1B}) \lambda_{1B} = \lambda_{2B} (1 + A_{2B}) + \left(1 - \bar{\xi}\right) \nu_{1B} \]

Externality

\[ - \left[ \frac{(\lambda_{1B} - \theta) [k_0 - k_{1B}] + \nu_{0B} (1 - \bar{\xi}) k_0}{\text{Risk sharing}} + \frac{\nu_{0B} (1 - \bar{\xi}) k_0}{\text{Collateral}} \right] F''(k_0 - k_{1B}) \]

\[ a_{0G} : \lambda_0 = \lambda_{1G} - \eta_{0G} + \nu_{0G} \quad a_{0B} : \lambda_0 = \lambda_{1B} - \eta_{0B} + \nu_{0B} \]
\[ a_{1G} : \lambda_{1G} = \lambda_{2G} - \eta_{1G} + \nu_{1G} \quad a_{1B} : \lambda_{1B} = \lambda_{2B} - \eta_{1B} + \nu_{1B} \]
\[ T_0 : \lambda_0 = \theta \]

25
The fact that the externality term enters symmetrically but with opposite sign in (31) and (32) clearly shows that the constrained planner’s optimal policy entails adjustments both in $k_0$ and in $k_{1B}$. Note also that the risk sharing term depends on the amount of fire sold capital $k_0 - k_{1B}$ but the collateral and margin terms are only a function of the amount of the capital $k_0$ held between $t = 0$ and $t = 1$.

Instead of discussing all the possible equilibrium combinations, I focus on the most relevant cases by making the following assumption.

**Assumption 2.** a) In equilibrium, the marginal utility of wealth in the bad state is strictly larger than at $t = 2$: $\lambda_1B > \lambda_2B$. b) The marginal utility at $t = 0$ is larger than the marginal utility in the good state at periods $t = 1$ and $t = 2$: $\lambda_0 > \lambda_1G > \lambda_2G$.

Assumption 2.a) implies that entrepreneurs value wealth more at $t = 1$ in the bad state than in the recovery at $t = 2$. This assumption can only arise in equilibrium if the entrepreneurs borrowing constraint at $t = 1$ in the bad state binds. Assumption 2.b) simply states that the entrepreneurs are always willing to hit their borrowing constraints in the good state at periods $t = 1$ and $t = 2$. This last assumption plays no role in the results and can be easily modified.

I consider two particular cases. These are analogous to cases 1 and 3 in proposition 1: a) $\lambda_1B > \lambda_0$, in this case entrepreneurs value wealth the most in the bad state at $t = 1$, so they are actively arranging insurance towards that state and the borrowing constraint does not bind; b) $\lambda_0 > \lambda_{1B}$, in this case entrepreneurs have very appealing investment opportunities ex-ante, so they hit their borrowing constraint towards the bad state.

Since the planner does not intervene in the good state, there is no reason to change policy there. Let’s set up instead the new problem for the constrained planner at $t = 1$ in the bad state. Note that $k_0$ is no longer a control variable and that $V_{M,1B}$ denotes the utility achieved by the outside consumers under the decentralized market conditional on being in the bad state at $t = 1$.

Observe that changes in $k_1$ will change the market value of the collateralized loan $a_{0B}$, generating the possibility that entrepreneurs decide not to repay. To simplify the analysis, I have opted for assuming that the entrepreneurs honor their inherited debt $a_{0B}$ even when $a_{0B} < -\left(1-\xi\right)p_{1B}k_0$ under the new allocation. I additionally impose the restriction that entrepreneurs always have sufficient resources to repay $a_{0B}$ and guarantee positive consumption.

$$\max_{k_{1B},a_{1B},C_{1B},C_{2B}} U(C_{1B}) + U(C_{2B}) + \theta \left[ e^C_{1} + e^C_{2} + (F(k_0 - k_{1B}) - F'(k_0 - k_{1B})[k_0 - k_{1B}] - V_{M,1B}^C \right]$$
C_1B + F' (k_0 - k_1B) k_1B = e_1B + [F' (k_0 - k_1B) + A_1B] k_0 + a_0B - a_1B \quad (\lambda_1B) \\
C_2B = e_2B + [1 + A_2B] k_1B + a_1B \quad (\lambda_2B) \\
a_1B \leq T_1B \quad (\eta_1B) \\
a_1B \geq - (1 - \xi) k_1B \quad (\nu_1B)

The optimality conditions for the constrained planner at \( t = 1 \) in the bad state are given by:

\begin{align*}
C_1B : U' (C_1B) = \lambda_1B & \quad C_2B : U' (C_2B) = \lambda_2B & \quad a_1B : \lambda_1B = \lambda_2B - \eta_1B + \nu_1B \\
\text{Externality} & \quad k_1B : F' (k_0 - k_1B) \lambda_1B = \lambda_2B (1 + A_2B) + \left( 1 - \xi \right) \nu_1B - \left[ (\lambda_1B - \theta) \left( k_0 - k_1B \right) \right] F'' (k_0 - k_1B) 
\end{align*}

It is obvious that the term corresponding to the borrowing constraint disappears from the new Euler equation for capital for the planner. The intuition is clear: higher prices during a fire sale increase ex-ante borrowing capacity, since they relax borrowing constraints by increasing the value of collateral or relaxing margins. In order to keep high prices during the fire sale, the planner commits ex-ante to an ex-post inefficiently high level of \( k_1B \). Once the fire sale state arrives, a planner has no incentive to introduce a distortion in the spot market: this results in a time inconsistency policy. Note that this kind of time inconsistency only arises when the borrowing constraints binds, that is when \( \lambda_0 > \lambda_1B \).

What about the risk-sharing term? We have to consider three different cases depending on whether we allow the planner to use various transfers:

1. **No lump sum transfers**: In this case, an analogous result to proposition 2 forces \( \theta = \lambda_1B \), which makes the externality term equal to zero and prevents the planner from finding a Pareto improvement. The planner decides to set a new \( k_1B \) different from the one set in the ex-ante problem. This value of \( k_1B \) corresponds to the decentralized allocation conditional on a given choice of \( k_0, C_0, a_0G, a_0B \) and \( T_0 \).

2. **A new lump sum transfer** \( T_1B \): allowing for this transfer would also imply \( \theta = \lambda_1B \), so the planner would implement the same allocation as that without any transfer. Intuitively, allowing for a transfer in the fire sale state is not helpful, since the planner was already able to redistribute resources in that state through price changes.

3. **An ex-post lump sum transfer** \( T_2B \): this transfer would imply \( \theta = \lambda_2B \). With this policy instrument, it is possible for the planner to find a Pareto improvement from a \( t = 1 \) perspective. If \( \lambda_1B > \lambda_0 > \lambda_2B \) the planner would decide to keep a higher \( k_1 \) than the one originally chosen.
and then introduce a large ex-post compensation to the outside consumers. If $\lambda_1 > \lambda_2 > \lambda_0$, the planner would choose a $k_1$ smaller than the one chosen at $t = 0$ but larger than the induced by the decentralized market at $t = 1$. When $\lambda_0 > \lambda_1 > \lambda_2$, the planner always chooses ex-post a level of $k_1$ larger than the optimal one ex-ante.

Hence, for all three types of transfers, the ex-ante constrained efficient policy is also time inconsistent due to risk sharing considerations. The intuition for the risk sharing mechanism when the constrained planner initially wants to have a small fire sale is the following: since the price is given by $p_{1B} = F'(k_0 - k_{1B})$, a planner can reduce the fire sale either by reducing the amount of $k_0$ held or by increasing the amount of $k_{1B}$. In order to minimize distortions, we expect the planner to adjust on both margins. Once the bad state arrives, unless the planner can use an ex-post transfer $T_{2B}$, the implicit transfer designed by the planner through a high $k_1$ is optimally undone by the decentralized trading between agents, so the planner finds it optimal to implement the market solution. When an ex-post transfer $T_{2B}$ is allowed, the constrained planner can again redistribute resources among agents by reducing the fire sale and compensating the outside consumers ex-post. In any case, the planner’s solution at $t = 1$ in the bad state will not coincide with the ex-ante optimal one. When the constrained planner would like to induce a large fire sale, the opposite logic applies. In this case, the constrained planner’s solution asks for a large $k_0$ and a small $k_{1B}$. Once the bad state at $t = 1$ occurs, there is no reason for entrepreneurs to choose less capital $k_{1B}$ than in the decentralized allocation, making the original policy time inconsistent.

We can summarize the results of this section in proposition 4:

**Proposition 4.** The optimal policy for a constrained efficient planner is time inconsistent. All three channels, risk sharing, collateral and margin, induce time inconsistency.

**Proof.** It follows from the discussion in the text. \qed

A natural step after identifying the time inconsistency problem is to characterize the time consistent constrained efficient policy. In this simply environment, the natural way to tackle this problem is by backwards induction: first, the planner solves for its optimal policy at period 1 and then takes into account its solution in the ex-ante $t = 0$ stage. In practical terms, the problem to solve is identical to the one discussed in section 4.2, with the exception that we need to add equations (33) and (34) as constraints in the planner problem. Unfortunately, the characterization of the time consistent policy need not be particularly intuitive. I thus relegate the solution of the time consistent problem to the online appendix. The main upshot of that exercise is that the time consistent ex-ante policy in the case that originally features overinvestment can imply a $k_0$ larger or smaller than the one in the decentralized market. The intuition for this result is that $k_1$ now depends endogenously on $k_0$, and a key quantity for the planner is the difference $k_0 - k_{1B}$. If $k_{1B}$ reacts more than proportionally with $k_0$, it may be optimal to allocate more capital ex-ante to the entrepreneurs, since $k_0 - k_{1B}$ will
be smaller and the fire sale reduced. In general, the final effects will depend on productivity $A_{1B}$ and $A_{2B}$, borrowing ability $\xi$ and the entrepreneurs’ attitude towards risk. The reverse reasoning can apply when the time $t = 0$ policy presents underinvestment.

Finally, how should we interpret these time inconsistency results in actual policy terms? The actual implications for time inconsistency of both the risk sharing and collateral externalities are extremely different. The reasoning behind the time inconsistency for the collateral channel, although natural inside the model, seems counterintuitive in reality: we don’t see policymakers reducing prices in crisis states because this high prices had already fulfilled their duty of allowing ex-ante borrowing. A compelling reason not to observe this policy behavior is that, by reducing prices in stress situations, the possibility of default, which I have assumed away, may rise substantially with negative consequences for other reasons unmodeled here.

From a policy perspective, the risk sharing mechanism has a much more appealing interpretation. Assume that we are in the most realistic situation, which corresponds to case 3 in proposition 1; entrepreneurs particularly value resources in the bad state. In this case, since the constrained planner wants to keep high prices in the bad state, he will decide to implement two different measures: a capital requirement to reduce initial investment $k_0$ and something equivalent to a lender of last resort policy to increase the level of $k_1$ (i.e., reducing the amount of fire sold capital) during the crisis. Once the crisis state actually arrives, if the planner can shift even more resources towards entrepreneurs by raising prices and compensating outside consumers with an ex-post transfer, he will gladly adopt this policy. This is the behavior that we usually observe in actual policymaking. If this transfer is not available, the planner cannot find Pareto improvements so he is forced to replicate the decentralized market solution, which implies a lower $k_1$ than the originally expected. In this case, an actual policymaker would probably disregard the Pareto criterion, downweighting the welfare of outside consumers and effectively avoiding the fall in prices.

### 6 Discussion of the results

Table 1 summarizes the main results of the paper:

<table>
<thead>
<tr>
<th>Cases</th>
<th>$\lambda_0$ and $\lambda_{1B}$</th>
<th>$k_0$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1: Maximum ex-ante borrowing</td>
<td>$\lambda_0 &gt; \lambda_{1B} &gt; \lambda_{1G}$</td>
<td>$k_{0\text{market}} \leq k_{0\text{planner}}$</td>
<td>Over- or Underinvestment</td>
</tr>
<tr>
<td>Case 2: Constraints do not bind locally</td>
<td>$\lambda_0 = \lambda_{1B} &gt; \lambda_{1G}$</td>
<td>$k_{0\text{market}} = k_{0\text{planner}}$</td>
<td>No Externalities</td>
</tr>
<tr>
<td>Case 3: Insufficient insurance</td>
<td>$\lambda_{1B} &gt; \lambda_0 &gt; \lambda_{1G}$</td>
<td>$k_{0\text{market}} &gt; k_{0\text{planner}}$</td>
<td>Overinvestment</td>
</tr>
</tbody>
</table>

Table 1: Summary of the results

In general, the collateral and margin channels always work in the direction of overinvestment, but the risk sharing channel can work in either direction. As long as the risk sharing channel is eliminated,
as it is when we assume a representative agent formulation, the planner’s solution implies a reduction in the amount of $k_0$ chosen. Observe that in the representative agent case, this reduction will trivially generate a Pareto improvement, even in the absence of transfers.

An important conclusion of this paper is that Pareto improvements are hard to find in fire sale environments when the planner properly accounts for heterogeneity and lacks the option of arranging ex-ante transfers. Previous work in this literature has emphasized the existence of Pareto improvements, but this paper shows that those only arise under very specific assumptions. However, the lack of Pareto improving policies does not mean that the externalities described here are not important or that capital regulations should be disregarded. My results highlight nonetheless that distributional considerations should be taken into account by the policymaker and that a cost-benefit analysis approach should be used; the discussion in Dreze and Stern (1987) about the Pareto criterion directly applies to this situation.

To gain deeper intuition, I present the constrained planner Euler equation for capital in the three period case where an ex-ante transfer is allowed and where both entrepreneurs and outside consumers are risk averse agents. This requires an endogenous choice of $k_{1B}$ different from $k_{1B} = 0$, as in the time inconsistency section. The equivalent expression to equation (29) is now (35). The multipliers of the entrepreneurs are represented with a superscript $E$ and those of the outside consumers with a superscript $C$.

\[ 1 = E \left[ \frac{\lambda_E}{\lambda_0} (p_{1s} + A_s) + \nu_{0s} \frac{1 - \xi (p_{1s})}{p_{1s}} \right] + \]

\[ + \pi_B \left[ \left( \frac{\lambda_{1B}^E}{\lambda_0^E} - \frac{\lambda_{1B}^C}{\lambda_0^C} \right) [k_0 - k_{1B}] + \nu_{0B} \frac{1 - \xi}{\lambda_0^C} \left( \xi' F' + \left( 1 - \xi \right) \right) k_0 \right] F'' \] (35)

The two key differences with respect to (29) are the following: a) the amount of asset fire sold $k_0 - k_{1B}$ is the key variable that determines the risk sharing externality but, since $k_{1B}$ plays no role in determining ex-ante borrowing capacity, only the amount of $k_0$ held affects both the margin and collateral channels; b) the difference between the ratio of intertemporal marginal rates of substitution of entrepreneurs with respect to outside consumers is the key term that determines the size of the risk sharing externality. (35) clearly shows that when markets are complete, i.e., $\frac{\lambda_E^B}{\lambda_0^B} = \frac{\lambda_C^B}{\lambda_0^B}$ and $\nu_{0B} = 0$, fire sale externalities do not matter. $\xi' F' = \frac{\Delta \xi}{\Delta p_{1s}/p_{1s}}$, which denotes the semi-elasticity of margins with respect to prices seems to be an interest object for future empirical work. Also $F''$, the sensitivity of prices, and $1 - \xi$, the pledgeability coefficient, are empirically relevant terms.

\footnote{See the online appendix for more details on the derivation.}
Another significant conclusion of this paper is that amplification mechanisms and welfare implications are conceptually two independent phenomena. This statement is not clear in the previous literature. The simple two period version of my model presents the three types of externalities but lacks any kind of amplification mechanism, since the entrepreneurs are always forced to sell all their capital in any event. When I introduce the three period formulation, both demand and supply of capital $k_0 - k_1B$ are downward sloping, creating amplification\textsuperscript{25}: a lower endowment in the bad state forces entrepreneurs to sell more capital, which reduces its price and makes entrepreneurs poorer, which makes them fire sell a larger amount, reducing prices even more, and so on. However, this amplification mechanism will also exist when credit constraints are not binding and markets are effectively complete in equilibrium (the equivalent situation to case 2 in proposition 1). We know that, in that case, the economy is efficient: these two cases show how associating amplification and welfare may be seriously misleading.

This decoupling result emphasizes that the existence of a “crisis” with very low prices does not justify a government intervention per se: a particular market failure has to be addressed by the policymaker. That said, the quantitative importance of the externalities will be larger if there are amplification mechanisms that generate deeper fire sales. Since $p_0 = 1$ is fixed in my model, no dynamic forward looking mechanisms such as those presented in Kiyotaki and Moore (1997) play a role. In that model, a binding price dependent borrowing constraint can simultaneously create amplification and externalities. This dynamic mechanism will generate amplification, but the sources of the externalities will be exactly the ones described in this paper.

Throughout the paper, I have adopted the assumption that entrepreneurs follow price taking behavior. If some entrepreneurs were aware of their non-infinitesimal size, they would internalize part of their effect on prices, modifying the effect of the externality. The larger the size of those agents, the greater the welfare transfer from the outside consumers. In my model, the $\theta = 0$ case is a monopolist that fully internalizes its impact on prices and doesn’t care about the welfare of others. Therefore, he would always set a smaller $k_0$ than in the price-taking case, hurting outside consumers. In general, other combinations of noncompetitive behavior could easily be explored inside my framework.

In appendix B, I discuss the case with a single risk-free asset, instead of two state contingent securities. That case is closer to reality than the main formulation of the paper in a particular aspect: in that situation, it is possible to have an entrepreneur who hits the borrowing constraint and who also believes that having resources in the crisis state is more valuable than at the initial period ($\lambda_{1B} > \lambda_0$). This is possible because now the entrepreneurs evaluate the benefits of riskless borrowing by considering the expected risk-adjusted returns, as in $\lambda_0 = \pi_G \lambda_{1G} + \pi_B \lambda_{1B} - \eta_0 + \nu_0$, instead of making that comparison state by state, as in $\lambda_0 = \lambda_{1B} - \eta_{0B} + \nu_{0B}$. Under this particular case, $\theta = 0$ is a monopolist that fully internalizes its impact on prices and doesn’t care about the welfare of others. Therefore, he would always set a smaller $k_0$ than in the price-taking case, hurting outside consumers. In general, other combinations of noncompetitive behavior could easily be explored inside my framework.

\textsuperscript{25}Section 6 in the online appendix shows a clarifying graph.
equilibrium configuration, as long as the entrepreneurs hit their borrowing constraints, there is always going to be overborrowing. This result is very intuitive: entrepreneurs do not internalize that the fire sale in the bad state depresses its borrowing capacity at the same time that it reduces their wealth in the state where they value it the most.

Two important lessons can be drawn from the comparison between the leading case in the paper and the one discussed in appendix B. First, the overall welfare effects due to pecuniary externalities crucially depend on the set of tradable assets. Hence, quantitative work must be particularly careful when specifying the set of trading opportunities. Second, the fact that agents lack state contingent insurance in fire sale states makes the desired overborrowing result more likely, since both the risk sharing and collateral/margin channels work in the same direction. This last result is of extreme importance, since there may be a stronger rationale for setting tight capital requirements when there are no opportunities to arrange state contingent borrowing.

Finally, this paper has focused only on the real effects of fire sales. An unexplored and fruitful avenue for future research would be to append a nominal side to the model and to study how a monetary authority’s influence on prices and wages\textsuperscript{26} interacts with all the channels described here.

7 Conclusion

This paper has shown how pecuniary externalities matter for fire sales. Two main mechanisms, price driven redistribution and borrowing capacity, through collateral value and margin tightening, determine the extent to which a constrained planner can achieve Pareto improvements with respect to the decentralized competitive market. Even though the ultimate root of all three channels is the presence of some form of market incompleteness, the way this incompleteness affects decentralized outcomes is patently different.

This paper is the first to identify the time inconsistency problem faced by a constrained planner in an environment with pecuniary externalities. All three channels are subject to the time inconsistency concern. I additionally emphasize how financial amplification mechanisms and pecuniary externalities are in general independent phenomena.

My results show that a planner cannot improve the decentralized outcome unless he decides to disregard distributional considerations or is allowed to use ex-ante transfers. For instance, the exclusive use of capital requirements is not enough to create Pareto improvements as long as the welfare of the buyers of fire sold assets is properly taken into account.

A necessary first step to guide the much needed financial regulatory reform is to identify the failures of the market mechanism: I hope that the tractability and scalability of my results help to focus future discussions about fire sales and pecuniary externalities in this context.

\textsuperscript{26}In this paper, I have not considered production and labor supply decisions or wage determination. In a larger scale model, those effects can potentially interact with the ones presented here.
APPENDIX

8 Appendix A: Proofs

8.1 Proof of proposition 2:

Proof. Without transfers, the indirect utility of the outside consumers depends directly on \( k_0 \). We can therefore write \( V_j' (k_0) = -\pi_B F'' (k_0') k_0' > 0 \) by using the envelope condition. This implies that in order to find a Pareto improvement, a necessary condition is \( k_{0\text{planner}} \geq k_{0\text{market}} \). However, we know that, at the decentralized equilibrium, a marginal change in \( k_0 \) modifies the indirect utility of the entrepreneurs by \( \pi_B \lambda_1 F'' (k_0) k_0 < 0 \) in any of the three equilibria discussed in proposition 1. This result can be easily derived using again the envelope condition. Therefore, if \( k_{0\text{planner}} > k_{0\text{market}} \), prices will be lower in the fire sale, necessarily hurting entrepreneurs. Thus, the only feasible policy for the constrained planner without transfers is to set \( k_{0\text{planner}} = k_{0\text{market}} \).

8.2 Proof of proposition 3:

Proof. I proceed by analyzing the analogous cases to each of the three cases described in proposition 1. In case 1, \( \lambda \frac{\mu}{\lambda_0} < 1, \nu_B > 0 \) and \( \eta_B = 0 \). It is useful to rewrite, by substituting \( \nu_B = \lambda_0 - \lambda_1 \), the externality term in (27) as \( \pi_B \left[ \left( \frac{\lambda \mu}{\lambda_0} - 1 \right) \left[ \xi + \xi' F' \right] F'' k_0 \right] \). If margins are fixed and \( \xi' = 0 \), or more generally when \( -\frac{\xi}{F'} < \xi' \), then we would expect the externality term to be negative. If this happens, then \( k_{0\text{market}} < k_{0\text{planner}} \), \( T_0 < 0 \) and the entrepreneurs underinvest in the decentralized market. If we constrain our solution to \( T_0 \geq 0 \), then the solution would imply \( k_{0\text{market}} = k_{0\text{planner}} \). When \( -\frac{\xi}{F'} > \xi' \), then \( k_{0\text{market}} > k_{0\text{planner}} \) and \( T_0 > 0 \). The case with \( \xi = 0 \) is the knife edge situation in which the risk sharing and collateral channels exactly cancel.

In case 2, no constraints bind towards the bad state, therefore, markets are locally complete, this is, \( \lambda \frac{\mu}{\lambda_0} = 1 \) and \( \nu_B = \eta_B = 0 \). Consequently, \( k_{0\text{market}} = k_{0\text{planner}} \) and \( T_0 = 0 \).

In case 3, \( \lambda \frac{\mu}{\lambda_0} > 1 \), \( \nu_B = 0 \), \( \eta_B > 0 \). This implies that \( k_{0\text{market}} > k_{0\text{planner}} \) and a positive transfer \( T_0 > 0 \) is implemented. This case is analogous to the leading case in Lorenzoni (2008).
9 Appendix B: The case of a single noncontingent asset

In this formulation, instead of having two state contingent securities, I assume that entrepreneurs can only use a risk-free bond. The rest of the notation and the assumptions are identical to main case discussed in the paper. Since this bond is traded by the outside consumers and they are risk neutral with discount factor of 1, both its price and return must be equal to 1. Using again an argument à la Hart and Moore (1994), I assume that the amount entrepreneurs can borrow ex-ante is limited by the maximum amount that can be recovered by the outside consumers in the worst case scenario $p_{1B}k_0$. I append once again the margin correction $(1 - \xi(p_{1B}))$ to the borrowing limit and appeal to section 2 of the paper to justify its relevance. Since $p_{1B}$ is always smaller than $p_{1G}$ by assumption, the borrowing constraint can be written as $a_0 \geq -\min\{(1 - \xi(p_{1s}))p_{1sk_0}\}$, where $\min\{(1 - \xi(p_{1s}))p_{1sk_0}\} = (1 - \xi(p_{1B}))p_{1B}k_0$. The decentralized version of this problem is then:

$$
\max_{k_0, a_0, C_0, C_{1G}, C_{1B}} U(C_0) + \pi_G U(C_{1G}) + \pi_B U(C_{1B})
$$

$$
C_0 + p_0k_0 = e_0 - q_0a_0 \quad (\lambda_0)
$$

$$
C_{1G} = e_{1G} + [p_{1G} + A_G]k_0 + a_0 \quad (\pi_G \lambda_{1G})
$$

$$
C_{1B} = e_{1B} + [p_{1B} + A_B]k_0 + a_0 \quad (\pi_B \lambda_{1B})
$$

$$
a_0 \leq \mathcal{L} \quad (\eta_0)
$$

$$
a_0 \geq -[1 - \xi(p_{1B})]p_{1B}k_0 \quad (\nu_0)
$$

With the following first order conditions for optimality in equilibrium:

$$
C_0 : U'(C_0) = \lambda_0 \quad C_{1G} : U'(C_{1G}) = \lambda_{1G} \quad C_{1B} : U'(C_{1B}) = \lambda_{1B}
$$

$$
a_0 : \lambda_0 = \mathbb{E} [\lambda_{1s}] - \eta_0 + \nu_0
$$

$$
k_0 : p_0\lambda_0 = \mathbb{E} [\lambda_{1s} (p_{1s} + A_s)] + \nu_0 [1 - \xi(F'(k_0))] F'(k_0)
$$

As described in the paper, depending on the value of $\lambda_0$, $\lambda_{1G}$ and $\lambda_{1B}$ there can be different type of equilibria. The most interesting case to consider here is the one with a binding borrowing constraint $\nu_0 > 0$. Note that with a single risk-free asset we can have a binding borrowing constraint at $t = 0$ with $\lambda_0 > \lambda_{1B}$ or with $\lambda_0 < \lambda_{1B}$. This last situation, when $\lambda_0 < \lambda_{1B}$, appears to be the most interesting one from an empirical viewpoint. In that case, the entrepreneurs value wealth the most in the bad state. Hence, if they had access to state contingent securities, they would arrange insurance for the bad state, however, they still hit their borrowing constraint ex-ante since they can only use an asset that is not state contingent.
Proceeding in the same way as in the section 4 in the paper, we can write the Euler equation for capital for the constrained planner as:

$$\lambda_0 = \mathbb{E} \left[ \lambda_{1s} (p_{1s} + A_{s}) \right] + \nu_0 (1 - \xi) F' + \left[ \pi_B (\lambda_{1B} - \theta) + \nu_0 (-\xi' F' + (1 - \xi)) \right] F'' k_0$$

The other optimality conditions remain identical to the ones in the decentralized market. If we allow for ex-ante transfers, $\lambda_0 = \theta$. The value of $\nu_0$, equal to $\lambda_0 - \lambda_{1B}$ in the case with state contingent assets, now becomes $\nu_0 = \lambda_0 - \pi_G \lambda_{1G} - \pi_B \lambda_{1B}$. Let’s then rewrite the externality term as:

$$\left[ \pi_B \left( \frac{\lambda_{1B}}{\lambda_0} - 1 \right) + \left( \pi_G \frac{\lambda_{1G}}{\lambda_0} + \pi_B \frac{\lambda_{1B}}{\lambda_0} - 1 \right) \left( -\xi' F' + (1 - \xi) \right) \right] F'' k_0$$

The importance of having a single contingent asset is that now we can have the case with $\nu_0 > 0$ and $\lambda_{1B} > \lambda_0$. In that situation, there is always going to be overinvestment. The intuition is the following: the entrepreneurs value having resources in the bad state more than ex-ante and than in the good state: $\lambda_{1B} > \lambda_0 > \lambda_{1G}$. Nonetheless, they borrow as much as they can and have to repay $a_0 = -[1 - \xi (p_{1B})] p_{1B} k_0$ in the bad state. Why? For this to happen we must have that $\lambda_0 > \pi_G \lambda_{1G} + \pi_B \lambda_{1B}$. This means that if carrying wealth in the risk-free asset towards the good state is not very valuable ($\lambda_{1G}$ low), and the bad state is somewhat not very likely ($\pi_B$ small), it may be ex-ante optimal to borrow as much as possible. There is no policy tradeoff in this case: a constrained planner will always set a $k_0^{\text{planner}}$ smaller than the one in the decentralized market, implementing for instance the appropriate level of capital requirements.

Figure 7 shows this case graphically:

![Figure 7: Equilibria with a single noncontingent asset.](image-url)
References


1 Understanding the mechanism: Risk Sharing Externality and Collateral Externality

These notes present a simpler version of the model than the one developed in the paper. In these notes there is a single state and there is no price-dependent margin mechanism. Most of the insights remain the same.

There are two dates, $t = 0, 1$. There are two agents: entrepreneurs (main actors) and outside consumers (they provide the downward sloping demand for assets in the fire sale). There are two goods: consumption goods (fruit) and capital goods (trees). Entrepreneurs are risk averse, maximize $U(C_0) + U(C_1)$ and have endowments of consumption good (fruit) of $e_0$ and $e_1$. They have to choose consumption (of fruit) $C_0$ and $C_1$, the amount of capital $k_0$ to hold between 0 and 1 (the number of trees to plant), and whether to save or borrow $a_0$ in a risk-free asset denominated in consumption good. Each unit of capital $k_0$ held by the entrepreneurs produces $A$ units of consumption goods at $t = 1$.

The consumption good is the numeraire, so $p_0$ and $p_1$ denote the prices of capital at $t = 0$ and $t = 1$ respectively. I assume that at $t = 0$, consumption goods and capital goods can be transformed on a one-to-one basis\(^1\), what pins down the price of capital (trees) at $t = 0$ to $p_0 = 1$ (net interest rate is 0). In period $t = 1$, they run a concave technology\(^2\) with capital that implies a price $p_1 = F'(k^C_1)$ and that, in equilibrium (since all capital is sold, yielding $k^C_1 = k_0$), generates $p_1 = F'(k_0)$. Any other assumptions that imply a downward sloping demand curve would also work (e.g., the buyers of the asset have limited risk-bearing capacity). Note that, in period $t = 0$, the outside consumers have no production technology, and they are only relevant to determine the amount $a_0$ of borrowing/lending.

The time subscript 0 or 1 denotes the period in which an endogenous variable is chosen.

---

\(^1\)We usually assume this in, for instance, the basic version of the neoclassical growth model.

\(^2\)They solve the problem $\max_{k^C_1} F(k^C_1) - p_1 k^C_1$, with FOC $p_1 = F'(k^C_1)$. 
1.1 Decentralized problem

Entrepreneurs solve the following problem:

$$\max_{k_0, a_0, C_0, C_1} \ U(C_0) + U(C_1)$$ \hspace{1cm} (1)

Subject to the following constraints:

$$C_0 + p_0 k_0 = e_0 - q_0 a_0 \quad (\lambda_0)$$  \hspace{1cm} (2)

$$C_1 = e_1 + [p_1 + A] k_0 + a_0 \quad (\lambda_1)$$  \hspace{1cm} (3)

$$a_0 \leq 0 \quad (\eta_0)$$ \hspace{1cm} (4)

$$a_0 \geq - (1 - \xi) p_1 k_0 \quad (\nu_0)$$ \hspace{1cm} (5)

(2) is the initial budget constraint. The entrepreneurs can consume \(C_0\), invest in \(k_0\) at price \(p_0\) (given my assumptions, \(p_0 = 1\)) with the endowment \(e_0\), save if \(a_0 > 0\) or borrow if \(a_0 < 0\).

(3) is the budget constraint in period 1. Since the world ends, all capital is sold \((k_1 = 0)\). The entrepreneurs consume their endowment, plus the fruit \(A k_0\) and sell the trees at the price \(p_1\). They repay their debts if they borrowed \((a_0 < 0)\) or consume the amount saved if \(a_0 > 0\).

(4) implies that agents cannot save. This equation can be motivated à la Holmstrom-Tirole by the lack of stores of value. The outside consumers that provide credit cannot commit and can steal everything. In the paper, I have the constraint \(a_0 \leq \mathcal{L}\) instead of \(a_0 \leq 0\). \(\mathcal{L}\) can be thought as the amount of stores of value in the economy.

(4) is a classic borrowing constraint a la Hart-Moore 94. The fruit can be stolen, so is not pledgeable. Only borrowing up to the market value of the tree \(p_1 k_0\) is allowed. I assume that a fraction \(\xi\) is lost when there is default, so the actual borrowing capacity is \((1 - \xi) p_1 k_0\).

Depending on which constraint (4) or (5) binds in equilibrium, we will have different welfare implications. The collateral channel is due to the fact that \(p_1\) appears on the constraint (5), so only when that constraint (5) binds that channel will appear. However, as long as any of the two credit constraints (4) or (5) binds, the risk sharing channel will play a role in equilibrium.

Note that, if instead of using \(a_0 \geq - (1 - \xi) p_1 k_0\) as borrowing constraint, we use \(a_0 \geq 0\) (this is, the trees cannot be pledged at all by the entrepreneurs), the collateral channel would immediately disappear. That would imply that both borrowing and lending are forbidden (i.e., in that case there are no asset markets). That is exactly the assumption that we would find in the literature on exogenous incomplete markets. In other words, exogenously specified incomplete markets can be understood as the particular case of borrowing constraints that are completely tight, \(a_0 = 0\) in my formulation.

The main problem of using the one-state version of the model is that in order for (4) to bind, we need to have entrepreneurs that save. With two states, the entrepreneurs could borrow at \(t = 0\) but still arrange insurance at the same time for the bad state at \(t = 1\).

1.1.1 Optimality conditions

Lagrangian:
\[
\mathcal{L} = U(C_0) + U(C_1) - \lambda_0 \left[ C_0 + p_0 k_0 + q_0 a_0 \right] - \lambda_1 \left[ C_1 - (p_1 + A) k_0 - a_0 \right] - \eta_0 a_0 + \nu_0 \left[ a_0 + (1 - \bar{\xi}) p_1 k_0 \right]
\]

First set of FOC’s:

\[
\begin{align*}
C_0 : & \ U''(C_0) = \lambda_0 \\
C_1 : & \ U''(C_1) = \lambda_1 \\
a_0 : & \ q_0 \lambda_0 = \lambda_1 - \eta_0 + \nu_0
\end{align*}
\]

Important FOC (Euler equation for capital):

\[
p_0 \lambda_0 = \lambda_1 \left( A + p_1 \right) + \nu_0 \left( 1 - \bar{\xi} \right) p_1
\]

1.1.2 Optimality conditions in equilibrium

Imposing the equilibrium conditions, the decentralized solution is fully characterized. This is:

\[
\begin{align*}
C_0 : & \ U''(C_0) = \lambda_0 \\
C_1 : & \ U''(C_1) = \lambda_1 \\
a_0 : & \ q_0 \lambda_0 = \lambda_1 - \eta_0 + \nu_0
\end{align*}
\]

And

\[
\lambda_0 = \lambda_1 \left( A + F'(k_0) \right) + \nu_0 \left( 1 - \bar{\xi} \right) F'(k_0)
\]

This last equation is the crucial one, and it will be compared to the planner’s solution.

1.2 Constrained planner’s problem

The planner solves the same problem as the decentralized agent, but he adds the constraint:

\[
\begin{align*}
\min_{k_0, a_0, C_0, C_1, T_0} \quad & e^C + T_0 + (F(k_0) - F'(k_0) k_0) \\
\text{subject to} \quad & \theta \left[ e^C + T_0 + (F(k_0) - F'(k_0) k_0) - V_C^M \right]\geq0
\end{align*}
\]

Utility for outside consumers under planner’s solution Utility for outside consumers in decentralized problem

Note that \( V_C^M \) is a given number, so this constraint can be thought of as a participation constraint for the outside consumers. Also notice how, given the risk neutrality of outside consumers, any borrowing and lending plays no role in their welfare. Let’s define \( \theta \) as the Lagrange multiplier of this constraint, that will bind in equilibrium. The planner maximizes:

\[
\max_{k_0, a_0, C_0, C_1, T_0} \quad U(C_0) + U(C_1) + \theta \left[ e^C + T_0 + (F(k_0) - F'(k_0) k_0) - V_C^M \right]
\]

3
$e^C$ is the large endowment of consumption good of outside consumers (we can think that $e^C = e_0^C + e_1^C$). $T_0$ is a possible ex-ante transfer. $T_0 > 0$ implies transfers for entrepreneurs to consumers, and $T_0 < 0$ vice versa. As discussed in the main text, if ex-ante transfers are not allowed, there are no feasible Pareto improvements.

Subject to the following constraints:

$$C_0 + k_0 + T_0 = e_0 - a_0 \quad (\lambda_0)$$
$$C_1 = e_1 + [F'(k_0) + A] k_0 + a_0 \quad (\lambda_1)$$
$$a_0 \leq 0 \quad (\eta_0)$$
$$a_0 \geq -(1 - \bar{\xi}) F'(k_0) k_0 \quad (\nu_0)$$

Note that the planner imposes the equilibrium prices before maximizing. Note also that this formulation implicit allows the planner to transfer resources of the consumption good. If we take literally the interpretation that consumption goods cannot be pledged, we have to admit that we are implicitly giving the planner an additional ability with respect to the market participants.

1.2.1 Optimality/Equilibrium conditions

Full Lagrangian:

$$L = U(C_0) + U(C_1) + \theta \left( e^C + T_0 + \pi_B (F(k_0) - F'(k_0) k_0) - V_M^C \right)$$
$$- \lambda_0 [C_0 + p_0 k_0 + q_0 a_0] - \lambda_1 [C_1 - [F'(k_0) + A] k_0 - a_0] - \eta_0 a_0 + \nu_0 \left[ a_0 + (1 - \bar{\xi}) F'(k_0) k_0 \right]$$

Standard FOC’s:

$$C_0 : U'(C_0) = \lambda_0$$
$$C_1 : U'(C_1) = \lambda_1$$
$$a_0 : \lambda_0 = \lambda_1 - \eta_0 + \nu_0$$

FOC for $T_0$:

$$\theta = \lambda_0$$

Intuition: in order to have an interior solution for the transfers, the marginal cost of moving wealth from entrepreneurs $\lambda_0$ must be equal to the marginal benefit of the entrepreneur $\theta$ (since they are risk neutral).

Important FOC (Euler equation for capital):

$$\lambda_0 = \lambda_1 (F'(k_0) + A) + (1 - \bar{\xi}) \nu_0 F''(k_0) + (\lambda_1 - \theta) k_0 F'''(k_0) + (1 - \bar{\xi}) \nu_0 k_0 F'''(k_0)$$

Risk-Sharing Collateral

Substituting $\theta = \lambda_0$:
Two interesting cases arise in equilibrium depending on which constraint binds:

- **Case 1: Only savings constraint binds (ν₀ = 0) - OVERINVESTMENT - ONLY RISK SHARING CHANNEL**

Entrepreneurs want to save from 0 to 1. They hit their savings constraint, there are no stores of value in the economy. Since they want to save, this means that they value wealth more at \(t = 1\) than at \(t = 0\), this is \(λ_1 > λ_0\).

The first order conditions in that case looks like:

\[
λ_0 = \frac{\text{Decentralized Term}}{\text{Risk-Sharing}} + \frac{\text{Externality}}{\text{Collateral}}
\]

\[
λ_0 = \lambda_1 (F'(k_0) + A) + (1 - \xi) ν_0 F'(k_0) + \left(\lambda_1 - λ_0\right) k_0 F''(k_0) + \left(1 - \xi\right) ν_0 k_0 F''(k_0)
\]

Since the externality term is negative, remember that \(F'' < 0\) and \(λ_1 - λ_0 > 0\), the planner wants to reduce investment ex-ante, there is **overinvestment**.

**Intuition:** entrepreneurs really value having resources in the future, but they lack stores of value. Since they do not internalize their effect in prices, they fire sale too much in \(t = 1\). A planner would reduce \(k_0\), reducing the fire sale and transferring wealth via this price change to the entrepreneurs in \(t = 1\). To compensate the outside consumers, who lose with this increase in price, a planner would given some positive transfer \(T_0\) ex-ante.

- **Case 2: Collateral constraint binds (ν₀ > 0) - OVER OR UNDERINVESTMENT - BOTH RISK SHARING AND COLLATERAL CHANNELS**

Entrepreneurs want to borrow at \(t = 0\) and repay at \(t = 1\). They hit the borrowing constraint due to their limited commitment. Since they want to borrow, this means that they value wealth more at \(t = 0\) than at \(t = 1\), this is \(λ_1 < λ_0\).

The first order conditions in that case looks like:

\[
λ_0 = \frac{\text{Decentralized Term}}{\text{Risk-Sharing}} + \frac{\text{Externality}}{\text{Collateral}}
\]

\[
λ_0 = \lambda_1 (F'(k_0) + A) + (1 - \xi) ν_0 p_1 + \left(\frac{\lambda_1 - λ_0}{ν_0 F''(k_0)}\right) k_0 F''(k_0)
\]

**Intuition:** the externality term has two parts.

1. Now the risk sharing term is positive, since \(λ_1 - λ_0 < 0\) and \(F''(k_0) < 0\). This term leads to **underinvestment**. This is natural, since agents are borrowing constrained and really would like to invest more ex-ante. The planner would give a transfer ex-ante to entrepreneurs and then compensate the outside consumers ex-post with a very large fire sale.

2. The collateral term is always negative, since \(ν_0 > 0\) and \(F''(k_0) < 0\). This term leads to **overinvestment**. The intuition here is that the entrepreneurs do not internalize the fact that by reducing prices they worsen the borrowing capacity of the other entrepreneurs, who will be able to borrow less.
In this particular case, we have that $\nu_0 = \lambda_0 - \lambda_1$. By substituting in (6):

$$\lambda_0 = \lambda_1 (F'(k_0) + A) + (1 - \xi) \nu_0 p_1 + (\lambda_1 - \lambda_0) [1 - (1 - \xi)] k_0 F''(k_0)$$

In this case, given the assumptions about traded assets, the externality term becomes always positive, that is, there would be underinvestment. As we can see in appendix B in the text, this conclusion crucially depend on the set of available assets traded.

Note: If I had allowed for a representative agent, or some kind of ex-post risk sharing mechanism between the parties, the risk sharing channel can be shut down. Most macro papers that start to use these ideas assume that, but this is worrisome. Doing this is shutting down the standard Walrasian mechanism. Aren’t the outside consumers buying cheap assets making a great deal? A different question is whether we (the planner) care about them.

### 2 Risk averse outside consumers

The model is identical to the one described above, with the exception that outside consumers are now risk averse with utility $V(\tilde{C}_0) + V(\tilde{C}_1)$, instead of being risk neutral. I use a tilde to denote choice variables of outside consumers.

Their budget constraints are:

$$\tilde{C}_0 = e_0^C + T_0 - q_0 \tilde{a}_0 \quad (\tilde{\lambda}_0)$$
$$\tilde{C}_1 = e_1^C + F(k_0) - p_1 B k_0 + \tilde{a}_0 \quad (\tilde{\lambda}_1)$$

Note that we don’t have to specify borrowing constraints for outside consumers.

The equilibrium conditions for outside consumers are given by:

$$\tilde{C}_0 : V'(\tilde{C}_0) = \tilde{\lambda}_0$$
$$\tilde{C}_1 : V'(\tilde{C}_1) = \tilde{\lambda}_1$$
$$\tilde{a}_0 : q_0 \tilde{\lambda}_0 = \tilde{\lambda}_1 \rightarrow q_0 = \frac{\tilde{\lambda}_1}{\tilde{\lambda}_0}$$

The conditions for the entrepreneurs are identical. The problem to solve for the constrained planner becomes now:

$$\max_{k_0, a_0, C_0, C_1, a_0, C_0, C_1, T_0} \left[ U(C_0) + U(C_1) + \theta \left[ V(e_0^C + T_0 - q_0 a_0) + V(e_1^C + F(k_0) - F'(k_0) k_0 + a_0) - V_M \right] \right]$$

Therefore, the constrained planner optimality condition for capital becomes:

$$\lambda_0 = \lambda_1 (F'(k_0) + A) + (1 - \xi) \nu_0 F'(k_0) + (\lambda_1 - \theta \lambda_1) k_0 F''(k_0) + (1 - \xi) \nu_0 k_0 F''(k_0)$$
Substituting the first order condition for $T_0$: $\lambda_0 = \theta \lambda_0$:

$$1 = \lambda_1 (F'(k_0) + A) + (1 - \bar{\xi}) \nu_0 F'(k_0) + \frac{\lambda_1}{\lambda_0} k_0 F''(k_0) + (1 - \bar{\xi}) \nu_0 k_0 F''(k_0)$$

A straightforward change of notation yields equation (36) in the paper.

3 Introducing a third period

The model is exactly the same as before, but now we introduce a third period $t = 2$. I assume that in period 2 capital can be eaten again, so $p_2 = 1$.

In this case outside consumers run the concave technology on the amount $k_0 - k_1$, this is the amount of capital sold by the entrepreneurs. I could assume that when $k_0 - k_1 < 0$, then the price is one (transforming consumption goods in capital). I will assume that we are in the case with $k_0 - k_1 > 0$, so the entrepreneurs are fire selling assets.

3.1 Decentralized problem

$$\max_{k_0, k_1, a_0, a_1, C_0, C_1, C_2} U(C_0) + U(C_1) + U(C_2) \quad (7)$$

Subject to the following constraints:

$$C_0 + p_0 k_0 = e_0 - q_0 a_0 \quad (\lambda_0)$$
$$C_1 + p_1 k_1 = e_1 + [p_1 + A_1] k_0 + a_0 - q_1 a_1 \quad (\lambda_1)$$
$$C_2 = e_2 + [p_2 + A_2] k_1 + a_1 \quad (\lambda_2)$$
$$a_0 \leq 0 \quad (\eta_0)$$
$$a_0 \geq - (1 - \bar{\xi}) p_1 k_0 \quad (\nu_0)$$
$$a_1 \leq 0 \quad (\eta_1)$$
$$a_1 \geq - (1 - \bar{\xi}) p_2 k_1 \quad (\nu_1)$$

In equilibrium, $q_0 = 1$, $q_1 = 1$, $p_0 = 1$, $p_1 = \min \{F'(k_0 - k_1), 1\}$, $p_2 = 1$ so the optimality conditions become:

$$C_0 : U'(C_0) = \lambda_0 \quad C_1 : U'(C_1) = \lambda_1 \quad C_2 : U'(C_2) = \lambda_2$$
$$k_0 : \lambda_0 = \lambda_1 (F'(k_0 - k_1) + A_1) + (1 - \bar{\xi}) \nu_0 F''(k_0 - k_1)$$
$$k_1 : p_1 \lambda_1 = \lambda_2 (1 + A_2) + (1 - \bar{\xi}) \nu_1$$
$$a_0 : \lambda_0 = \lambda_1 - \eta_0 + \nu_0 \quad a_1 : \lambda_1 = \lambda_2 - \eta_1 + \nu_1$$
3.2 Constrained planner’s problem

\[
\max_{k_0, k_1, a_0, a_1, C_0, C_1, C_2, T_0} U(C_0) + U(C_1) + U(C_2) + \theta \left[ e^C + T_0 + (F(k_0 - k_1) - F'(k_0 - k_1)[k_0 - k_1]) - V^C_M \right]
\]

\[
C_0 + k_0 + T_0 = e_0 - a_0 \quad (\lambda_0)
\]
\[
C_1 + F'(k_0 - k_1)k_1 = e_1 + [F'(k_0 - k_1) + A_1]k_0 + a_0 - a_1 \quad (\lambda_1)
\]
\[
C_2 = e_2 + [1 + A_2]k_1 + a_1 \quad (\lambda_2)
\]
\[
a_0 \leq 0 \quad (\eta_0)
\]
\[
a_0 \geq -(1 - \bar{\xi})F'(k_0 - k_1)k_0 \quad (\nu_0)
\]
\[
a_1 \leq 0 \quad (\eta_1)
\]
\[
a_1 \geq -(1 - \bar{\xi})k_1 \quad (\nu_1)
\]

With optimality conditions:

\[
C_0 : U'(C_0) = \lambda_0 \quad C_1 : U'(C_1) = \lambda_1 \quad C_2 : U'(C_2) = \lambda_2
\]

Decentralized Term
\[
k_0 : \lambda_0 = \lambda_1 (F'(k_0 - k_1) + A_1) + (1 - \bar{\xi})\nu_0 F'(k_0 - k_1) + ((\lambda_1 - \theta)[k_0 - k_1] + (1 - \bar{\xi})\nu_0 k_0) F''(k_0 - k_1)
\]

Externality
\[
k_1 : F'(k_0 - k_1) \lambda_1 = \lambda_2 (1 + A_2) + (1 - \bar{\xi})\nu_1 - [(\lambda_1 - \theta)[k_0 - k_1] + (1 - \bar{\xi})\nu_0 k_0] F''(k_0 - k_1)
\]

Decentralized Term
\[
a_0 : \lambda_0 = \lambda_1 - \eta_0 + \nu_0 \quad a_1 : \lambda_1 = \lambda_2 - \eta_1 + \nu_1
\]

Externality
\[
T_0 : \lambda_0 = \theta
\]

3.3 Time Inconsistency

Problem that the planner solves at \( t = 1 \).

\[
\max_{k_1, a_1, C_1, C_2} U(C_1) + U(C_2) + \theta \left[ e^C + \left( F(\bar{k}_0 - k_1) - F'(\bar{k}_0 - k_1)[\bar{k}_0 - k_1] \right) - V^C_M \right]
\]

\[
C_1 + F'(\bar{k}_0 - k_1)k_1 + T_1 = e_1 + [F'(\bar{k}_0 - k_1) + A_1]\bar{k}_0 + a_0 - a_1 \quad (\lambda_1)
\]
\[
C_2 = e_2 + [1 + A_2]k_1 + a_1 \quad (\lambda_2)
\]
\[
a_1 \leq 0 \quad (\eta_1)
\]
\[
a_1 \geq -(1 - \bar{\xi})k_1 \quad (\nu_1)
\]

With optimality conditions:
\[ C_1 : \frac{dU}{dC_1} = \lambda_1 \]
\[ C_2 : \frac{dU}{dC_2} = \lambda_2 \]

Decentralized Term
\[ k_1 : F'(k_0 - k_1) \lambda_1 = \frac{\lambda_2 (1 + A_2) + \nu_1}{[\lambda_1 - \theta] (k_0 - k_1)} \]

Externality
\[ a_1 : \lambda_1 = \lambda_2 - \eta_1 + \nu_1 \]

Note that the collateral externality term clearly disappears. Note also that given \( k_0 \), now fixed, a planner cannot induce a new constrained Pareto improvement, even if he had the chance to use transfers in period \( t = 1 \). If he had the chance to arrange ex-post redistribution, at \( t = 2 \) there may be a new possible improvement, but there is still time inconsistency. The agent that was supposed to be compensated by the price redistribution will lose.

4 Time consistent policy

This section of the online appendix analyzes the same case as in the paper, with 3 periods and 2 states. It characterizes the time consistent problem for a constrained planner. For simplicity, I only discuss the case of a planner who can just use an ex-ante transfer \( T_0 \). This assumption implies that at \( t = 1 \) in the bad state, the planner will not honor the predetermined \( k_1 \); he will replicate the same solution as the market. We can combine the optimality conditions from a \( t = 1 \) perspective to yield equation (15) and then find the ex-ante time consistent solution. If we allow the constrained planner to use an ex-post transfer \( T_2 \), there would be additional term taking that into account, but the reasoning would be identical.

From equations (33) and (34) in the text, the additional constraint that the planner faces at \( t = 0 \) is:

\[ [F'(k_0 - k_1) - (1 - \bar{\xi})] \lambda_{1B} = \lambda_{2B} (A_{2B} + 1 - (1 - \bar{\xi})) \quad (\pi_B \psi) \]

Or substituting the marginal value of wealth:

\[ [F'(k_0 - k_1) - (1 - \bar{\xi})] U'(e_{1B} + A_{1B}k_0 + F'(k_0 - k_1)k_1 + a_{0B} - a_{1B}) - U'(e_{2B} + [A_{2B} + 1]k_{1B} + a_{1B}) (A_{2B} + 1 - (1 - \bar{\xi})) = 0 \quad (\pi_B \psi) \]

Therefore, the ex-ante problem for the constrained planner yields the following optimality conditions:

\[ C_0 : \frac{dU}{dC_0} = \lambda_0 \]
\[ C_{1s} : \frac{dU}{dC_1} = \lambda_{1s}, \text{ for } s = G, B \]
\[ C_{2s} : \frac{dU}{dC_2} = \lambda_{2s}, \text{ for } s = G, B \]
\[ a_{0G} : \lambda_0 = \lambda_1 - \eta_0 + \nu_0 \quad a_{1G} : \lambda_1 = \lambda_2 - \eta_1 + \nu_1 \]

\[ a_{0B} : \lambda_0 = \lambda_1 - \eta_0 + \nu_0 + \psi \left[ F'(k_0 - k_1B) - (1 - \bar{\xi}) \right] U''(C_{1B}) \]

\[ a_{1B} : \lambda_1 = \lambda_2 - \eta_1 + \nu_1 + \psi \left[ - \left[ F'(k_0 - k_1B) - (1 - \bar{\xi}) \right] U''(C_{1B}) - (A_{2B} + 1 - (1 - \bar{\xi}) \right] U''(C_{2B}) \]

\[ T_0 : \lambda_0 = \theta \]

\[ k_0 : \lambda_0 = \mathbb{E} [\lambda_1 (A_x + p_{1s})] + \mathbb{E} [\nu_0 p_{1s}] + \pi_B \left[ \left( (\lambda_1 - \theta) [k_0 - k_1B] + \nu_0 (1 - \bar{\xi}) k_0 \right) F''(k_0 - k_1B) \right] \]

\[ + \pi_B \psi \left( F''(\cdot) \lambda_1B + \left[ F'(\cdot) - (1 - \bar{\xi}) \right] U''(C_{1B}) \left[ F''(\cdot) [k_0 - k_1B] + F'(\cdot) \right] - \left[ A_{2B} + 1 - (1 - \bar{\xi}) \right] U''(C_{2B}) \left[ A_{2B} + 1 \right] \right] \]

By substituting \( \nu_{1B} \) from (17) into (19) and imposing (15), we can find that:

\[ \left[ \left( (\lambda_1 - \theta) [k_0 - k_1B] + \nu_0 (1 - \bar{\xi}) k_0 \right) F''(k_0 - k_1B) \right] = \]

\[ \psi \left( (-F''(\cdot) \lambda_1B + \left[ F'(\cdot) - (1 - \bar{\xi}) \right] U''(C_{1B}) \left[ F''(\cdot) [k_0 - k_1B] + F'(\cdot) \right] - \left[ A_{2B} + 1 - (1 - \bar{\xi}) \right] U''(C_{2B}) \left[ A_{2B} + 1 \right] \right] \]

Thus:

\[ \left[ \left( (\lambda_1 - \theta) [k_0 - k_1B] + \nu_0 (1 - \bar{\xi}) k_0 \right) F''(k_0 - k_1B) \right] = \]

\[ \psi \left( (-F''(\cdot) \lambda_1B + \left[ F'(\cdot) - (1 - \bar{\xi}) \right] U''(C_{1B}) \left[ F''(\cdot) [k_0 - k_1B] + F'(\cdot) - (1 - \bar{\xi}) \right] - \left[ A_{2B} + 1 - (1 - \bar{\xi}) \right] U''(C_{2B}) A_{2B} \right) \]

And then substituting back into (18):

\[ \lambda_0 = \mathbb{E} [\lambda_1 (A_x + p_{1s})] + \mathbb{E} [\nu_0 p_{1s}] + \pi_B \psi \left( \left[ F'(\cdot) - (1 - \bar{\xi}) \right] U''(C_{1B}) \left[ (1 - \bar{\xi}) + A_{1B} \right] - \left[ A_{2B} + 1 - (1 - \bar{\xi}) \right] U''(C_{2B}) A_{2B} \right) \]

Substituting now (15):
\[\lambda_0 = \mathbb{E} [\lambda_{1s} (A_s + p_{1s})] + \mathbb{E} [(\lambda_0 - \lambda_{1s} + \eta_{1s}) p_{1s}] + \]
\[+ \pi_B \psi [A_{2B} + 1 - (1 - \xi)] U' (C_{2B}) \left( \frac{U'' (C_{1B})}{U'(C_{1B})} [(1 - \xi) + A_{1B}] - \frac{U'' (C_{2B})}{U'(C_{2B})} A_{2B} \right) \]  

(20)

If \( \nu_{0B} \) is positive, we need to substitute \( \nu_{0B} \) from (16) into (21) to have the appropriate comparison. This yields:

\[\lambda_0 = \mathbb{E} [\lambda_{1s} (A_s + p_{1s})] + \mathbb{E} [(\lambda_0 - \lambda_{1s}) p_{1s}] + \]
\[+ \pi_B \psi \left( [F' (\cdot) - (1 - \xi)] U'' (C_{1B}) [(1 - \xi) + A_{1B} - F' (\cdot)] - [A_{2B} + 1 - (1 - \xi)] U'' (C_{2B}) A_{2B} \right) \]

Substituting back (15):

\[\lambda_0 = \mathbb{E} [\lambda_{1s} (A_s + p_{1s})] + \mathbb{E} [(\lambda_0 - \lambda_{1s}) p_{1s}] + \]
\[+ \pi_B \psi [A_{2B} + 1 - (1 - \xi)] U' (C_{2B}) \left( \frac{U'' (C_{1B})}{U'(C_{1B})} [(1 - \xi) + A_{1B} - F' (\cdot)] - \frac{U'' (C_{2B})}{U'(C_{2B})} A_{2B} \right) \]  

(21)

When we are in the overinvestment case, \( \psi \) is negative and positive otherwise. Hence, from either (20) or (21) it is not obvious whether the planner is going to choose a \( k_0 \) larger or smaller than in the decentralized market. The intuition for this results is that \( k_1 \) is a function of \( k_0 \), and the correction term for the constrained planner depends on \( k_0 - k_1 (k_0) \). Even if we are in the situation with overinvestment, it may be the case that an increase in \( k_0 \) increases endogenously \( k_1 (\cdot) \) more than proportionally, reducing the fire sale and creating the desired insurance effect through prices.

5 Uniqueness

I show how to find sufficient conditions for uniqueness in the decentralized problem. For simplicity, I focus on the simplest case described in this appendix, with only two periods, one state and not price dependent margins. Similar results can be find in the more general cases.

\[C_0 + p_0 k_0 = e_0 - q_0 a_0 \quad (\lambda_0) \]  
\[C_1 = e_1 + [p_1 + A] k_0 + a_0 \quad (\lambda_1) \]  
\[a_0 \leq 0 \quad (\eta_0) \]  
\[a_0 \geq -(1 - \xi) p_1 k_0 \quad (\nu_0) \]  

(22)
\( (23) \)
\( (24) \)
\( (25) \)

Assuming that \( a_0 = -(1 - \xi) p_1 k_0 \quad (\nu_0) \) binds, the worst case in principle to show uniqueness, the equilibrium capital choice \( k_0 \) is determined by:

\[U' (e_0 + (1 - \xi) F' (k_0) k_0 - k_0) (1 - (1 - \xi) F' (k_0)) = \]
\[= U' (e_1 + [F' (k_0) + A] k_0 - (1 - \xi) p_1 k_0) (A + F' (k_0) - (1 - \xi) F' (k_0)) \]  

(26)
The left hand side derivative with respect to $k_0$ is given by:

$$U''(\cdot) (1 - (1 - \bar{\xi}) F''(k_0)) \left( (1 - \bar{\xi}) \left| F''(k_0) k_0 + F'(k_0) \right| - 1 \right) - U'(\cdot) (1 - \bar{\xi}) F'''(k_0) > 0$$

This shows that the left hand side in (26) is increasing in $k_0$, as depicted in figure 5 in the text.

The right hand side derivative with respect to $k_0$ in (26) is given by:

$$U''(\cdot) \left( A + \bar{\xi} F'(k_0) \right) \left( A + \bar{\xi} \left[ F'''(k_0) k_0 + F'(k_0) \right] \right) + U'(\cdot) \left( \bar{\xi} F''(k_0) \right) < 0$$

This term must be positive

This shows that a sufficient condition for the right hand side in (26) to be negative is that:

$$(A + \bar{\xi} \left[ F'''(k_0) k_0 + F'(k_0) \right]) > 0$$

In practical terms this implies that $F''$ cannot be too large, this is, small changes in the amount of fire sold assets cannot depress prices too quickly. Once $k_0$ is determined, the rest of the variables are uniquely determined too.

An analogous approach can be used to derive conditions for the constrained planner’s problem. Using the same reasoning, we can also provide sufficient conditions for uniqueness for the more general cases discussed in the paper.

6 Illustrating the lack of amplification

Figure 1 clearly shows how the two period formulation lacks any kind of amplification in the bad state. When the amount of capital sold is fixed in the bad state, as the inelastic supply of capital $k_0$ represents, there is no room for amplification. However, when the supply of capital by the entrepreneurs is decreasing in price, there is room for spirals and cycles in which low prices induce more sales of assets, which depress prices and so on. Observe that I have assumed away multiplicity problems, which can potentially exist in these environments. My analysis can be easily adapted to those situations by using local arguments.
Capital sold

Supply of capital

$p_{1B} = F'(\cdot)$

Figure 1: Showing the lack of amplification.