Sequential Equilibria of Multi-Stage Games with Infinite Sets of Types and Actions

by

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Abstract: We formulate a definition of basic sequential equilibrium for multi-stage games with infinite type sets and infinite action sets, and we prove its general existence. We then explore several difficulties of this basic concept and, in light of them, propose the more restrictive definition of essential sequential equilibrium while maintaining general existence.

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**Goal:** formulate a definition of *sequential equilibrium* for multi-stage games with infinite type sets and infinite action sets, and prove general existence. Sequential equilibria were defined for finite games by Kreps-Wilson 1982, but rigorously defined extensions to infinite games have been lacking. Various formulations of "perfect bayesian eqm" (defined for *finite* games in Fudenberg-Tirole 1991) have been used for infinite games. No general existence. Harris-Stinchcombe-Zame 2000 explored definitions with nonstandard analysis.

It is well understood that sequential equilibria of an infinite game can be defined by taking limits of sequential equilibria of finite games that approximate it. The problem is to define what finite games are good approximations. It is easy to define sequences of finite games that seem to be converging to the infinite game (in some sense) but have limits of equilibria that seem wrong. We must try to define a class of finite approximations that yield limit-equilibria which include reasonable equilibria and exclude unreasonable equilibria. Here we present the best definitions that we have been able to find, but of course this depends on intuitive judgments about what is "reasonable". Others should explore alternative definitions.
Dynamic multi-stage games $\Gamma=($Θ,N,K,A,T,p,τ,v$)$

Θ = \{initial state space\}, i \in N = \{players\}, finite set.
k ∈ \{1,...,K\} periods of the game.
Let L = {(i,k) | i ∈ N, k ∈ {1,...,K}} = \{dated players\}. We write ik for (i,k).
A_{i,k} = \{possible actions for player i at period k\}; history dependence through payoffs.
T_{i,k} = \{possible informational types for player i at period k\}, disjoint sets.
Algebras (closed under finite $\cap$ and complements) of measurable subsets are specified for Θ and $T_{i,k}$, including all one-point sets as measurable.
If Θ is a finite set, then all subsets of Θ and $T_{i,k}$ are measurable.
There is a finitely additive probability measure $p$ on the measurable subsets of Θ.
$A = \times_{h \leq K} \times_{i \in N} A_{i,h} = \{possible sequences of actions in the whole game\}.$
The subscript, $< k$, denotes the projection onto periods before k. For example,
$A_{< k} = \times_{h < k} \times_{i \in N} A_{i,h} = \{possible action sequences before period k\}$  ($A_{< 1} = \{∅\}$),
and for $a ∈ A$, $a_{< k} = \times_{h < k} \times_{i \in N} a_{i,h}$ is the partial sequence of actions before period k.
Any player i's information at any period k is specified by a type function $\tau_{i,k}: Θ × A_{< k} \rightarrow T_{i,k}$ such that, $\forall a ∈ A$, $\tau_{i,k}(\theta,a_{< k})$ is a measurable function of $\theta$.
Assume perfect recall: $\forall i k \in L$, $\forall m < k$, $\exists \rho_{ikm}: T_{i,k} \rightarrow T_{i,m} × A_{i,m}$ such that
$\rho_{ikm}(\tau_{i,k}(\theta,a_{< k})) = (\tau_{i,m}(\theta,a_{< m}),a_{i,m})$, $\forall \theta ∈ Θ$, $\forall a ∈ A$, and
$\{t_{ik} | \rho_{ikm}(t_{ik}) \in T_{i,m} × \{a_{i,m}\}\}$ is measurable in $T_{i,k}$, $\forall$ measurable $R_{i,m} ⊆ T_{i,m}$, $\forall a_{i,m} ∈ A_{i,m}$.
Each player i has a bounded utility function $v_i: Θ × A \rightarrow \mathbb{R}$ such that
$v_i(\theta,a)$ is a measurable function of $\theta ∈ Θ$, $\forall a ∈ A$. Bound $|v_i(\theta,a)| ≤ Ω$, $\forall (i,\theta,a)$. 
Example. (Kuhn) Consider a zero-sum game in which player 1 chooses a number $a_1$ in $[0,1]$, and player 2 chooses a continuous function, $f$, from $[0,1]$ into itself whose Lebesgue integral must be $1/2$. Player 1’s payoff is $f(a_1)$. Player 1 can guarantee $1/2$ by choosing $a_1$ uniformly from $[0,1]$ and player 2 can guarantee $1/2$ by choosing $f(x) = 1/2$ for all $x$.

But in any finite approximation in which player 2 can choose a function that is zero at each of player 1’s finitely many available actions, player 1’s equilibrium payoff is 0.

One solution is to replace ordinary action sets with mixed action sets. Algebras (closed under finite $\cap$ and complements) of measurable subsets are specified also for each $A_{ik}$, including all one-point sets as measurable. Each $\tau_{ik}(\theta,a_{<k})$ and each $v_i(\theta,a)$ is assumed jointly measurable in $(\theta,a)$.

Let $\Theta = \Theta \times (\times_{ik} [0,1])$. In the state $\tilde{\theta} = (\theta,(\tilde{\theta}_{ik})_{ik \in L})$, nature draws $\theta$ from the given $p$, and, for each $ik$, draws $\tilde{\theta}_{ik} \in [0,1]$ independently from Lebesgue measure.

This defines a new distribution $\tilde{p}$ on $\tilde{\Theta}$. No player observes any of the new $\tilde{\theta}_{ik}$.

Let $\tilde{A}_{ik} = \{\text{measurable maps } \tilde{a}_{ik} : [0,1] \to A_{ik} \}$ be player $ik$’s set of mixed actions. Then $\tilde{v}_i(\tilde{\theta},\tilde{a}) = v_i(\theta,\tilde{a}(\tilde{\theta}))$ and $\tilde{\tau}_{ik}(\tilde{\theta},\tilde{a}_{<k}) = (\tau_{ik}(\theta,\tilde{a}(\tilde{\theta})_{<k}), \tilde{a}_{i,<k})$ are measurable in $\tilde{\theta}$ for each profile of mixed actions $\tilde{a}$.

All our analysis could be done for this model with $\tilde{A},\tilde{\Theta},\tilde{p}$ instead of $A,\Theta,p$. 
Problems of spurious signaling in approximating games

Example. Nature chooses $\theta \in \{-1,1\}$, prob. $\frac{1}{2}$ each. Player 1 observes $t_1 = \theta$ and chooses $a_1 \in [-1,1]$. Player 2 observes $t_2 = \theta \cdot a_1$ and chooses $a_2 \in \{-1,1\}$. Payoffs are as follows.

<table>
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<tr>
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<th>$a_2 = -1$</th>
<th>$a_2 = 1$</th>
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<tbody>
<tr>
<td>$\theta = -1$</td>
<td>(0,2)</td>
<td>(1,0)</td>
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<tr>
<td>$\theta = 1$</td>
<td>(0,0)</td>
<td>(1,1)</td>
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No observation $t_2 \in [-1,1]$ should lead player 2 to choose $a_2 = 1$ since player 1 always prefers $a_2 = 1$ and the set of signals player 1 can send does not depend on $\theta$.

But any approximation where 1’s finitely many actions are, e.g., all rational when $t_1 = 1$ and are all irrational when $t_1 = -1$ reveals $\theta$, so $a_2 = 1$ when $t_2$ is any rational number.

To avoid such spurious signaling, the finite set of actions available to a player should not depend on his informational type.

But now consider any finite approximation in which player 1’s finite action set does not depend on his type and in which the action $a_1 = x > 0$ is available to player 1 but action $a_1 = -x$ is not. Then $t_2 = x$ implies $\theta = 1$, leading player 2 to choose $a_2 = 1$.

To avoid also the latter spurious signaling, we should limit player 2’s ability to distinguish her types and allow player 1 any finite subset of his actions. So, we will coarsen the players’ information by finitely partitioning each of their type spaces, $T_{ik}$, and we will expand the players’ finite action sets to fill in the action spaces $A_{ik}$ faster than the type-space partitions become arbitrarily fine.
Problems of spurious knowledge in approximating games
Some approximations of the state space $\Theta$ can change the information structure and give players information that they do not have in the original game.

Example. $\Theta$ includes two independent uniform [0,1] random variables: the first (1's cost to sell some object) is observed only by player 1, the second (2's value of buying this object) is observed only by player 2. For any integer $m > 0$, we might finitely approximate this game by one in which the state space includes $10^{2m}$ equally likely pairs of numbers: the first number, observed by 1, ranges over the $10^m$ m-digit decimals in [0,1]; the second, observed by 2, is a 2m-digit decimal in which the first m digits range over all m-digit decimals but the last m digits repeat the first number in the pair. As $m \to \infty$, these pairs uniformly fill the unit square in the state space $\Theta$, but they represent games in which player 2 knows 1's type.

With finite partitions of the $T_{ik}$ we can keep the original state space $\Theta$ and probability distribution $p$ in all finite approximations, avoiding the above information distortion. By partitioning each $T_{ik}$ separately, we avoid giving spurious knowledge to any player.

In some games, the ability to choose some special action in $A_{ik}$ or the ability to distinguish some special subsets of $T_{ik}$ may be particularly important. So we consider convergence of a net of approximations (not just a sequence) that are indexed on all finite subsets of $A_{ik}$ and all finite partitions of $T_{ik}$. So, any finite collection of actions and any finite partition of types are available when considering the properties of the limits.
Observable events and action approximations
For any \( a \in A \) and any measurable \( R_{ik} \subseteq T_{ik} \), let \( P(R_{ik}|a) = p(\{ \theta \mid \tau_{ik}(\theta, a_{<k}) \in R_{ik} \}) \).
(For \( k=1 \), we could write \( P(R_{i1}) \) for \( P(R_{i1}|a) \), ignoring the trivial \( a_{<1} = \emptyset \).)

The set of observable events for \( i \) at \( k \) that can have positive probability is
\[ \mathcal{E}_{ik} = \{ R_{ik} \subseteq T_{ik} \mid R_{ik} \text{ is measurable and } \exists a \in A \text{ such that } P(R_{ik}|a) > 0 \} . \]

Let \( \mathcal{E} = \bigcup_{ik \in L} \mathcal{E}_{ik} \) (a disjoint union) denote the set of all events that can be observed with positive probability by some dated player.

We extend this notation to describe observable events that can have positive probability when players are restricted to actions in some subsets of the \( A_{ik} \).
An action approximation is any \( C = \times_{ik \in L} C_{ik} \) such that each \( C_{ik} \) is a nonempty finite subset of \( A_{ik} \), and so \( C \subseteq A \).

For any action approximation \( C \subseteq A \), let
\[ \mathcal{E}_{ik}(C) = \{ R_{ik} \subseteq T_{ik} \mid R_{ik} \text{ is measurable and } \exists c \in C \text{ such that } P(R_{ik}|c) > 0 \} . \]

Let \( \mathcal{E}(C) = \bigcup_{ik \in L} \mathcal{E}_{ik}(C) \) denote the set of all events that can be observed with positive probability by some dated player when all players use actions in \( C \).
\( \mathcal{E} \) is the union of all \( \mathcal{E}(C) \) over all finite approximations \( C \).

Action approximations are partially ordered by inclusion.
If \( C \supseteq C^o \) then \( C \) is a better approximation than \( C^o \) to the true action sets \( A \).

Fact. If \( C \supseteq C^o \) then \( \mathcal{E}(C) \supseteq \mathcal{E}(C^o) \).
Finite approximations of the game

An information approximation is any $\Pi = \times_{ik \in L} \Pi_{ik}$ such that each $\Pi_{ik}$ is a finite partition of measurable subsets of $T_{ik}$. (So elements of $\Pi_{ik}$ are disjoint measurable sets with union $T_{ik}$.)

Information approximations are partially ordered by the fineness of their partitions. Say that $\Pi = \times_{ik \in L} \Pi_{ik}$ is finer than $\Pi^0 = \times_{ik \in L} \Pi^0_{ik}$ if $\Pi_{ik}$ is a finer partition of $T_{ik}$ than $\Pi^0_{ik}$ $\forall ik \in L$. (If $\Pi$ is finer than $\Pi^0$, then $\Pi$ is a better approximation than $\Pi^0$ to the true type sets $T$.)

An information approximation $\Pi$ and an action approximation $C$ together define a finite approximation $(\Pi, C)$ of the game $\Gamma$.

After any history $(\theta, c_{<k})$ in the approximating game $(\Pi, C)$, each player $i_k$ is informed of the $\pi_{ik} \in \Pi_{ik}$ that contains $\tau_{ik}(\theta, c_{<k})$ and is also informed of his past actions $c_{i(<k)}$. The latter ensures that action-recall is not violated in $(\Pi, C)$ when we expand $C$ to fill in the action space $A$ for any fixed information approximation $\Pi$.

Let $S_{ik} = \{(\pi_{ik}, c_{i(<k)}) \in \Pi_{ik} \times C_{i(<k)} : \exists (\theta, c') \in \Theta \times C \text{ s.t. } \tau_{ik}(\theta, c'_{<k}) \in \pi_{ik} \text{ and } c'_{i(<k)} = c_{i(<k)}\}$ denote the finite set of possible types $s_{ik}$ of dated player $i_k$ in the finite approximation $(\Pi, C)$.

Fix $\Pi$. The approximation $(\Pi, C)$ will satisfy perfect recall $\forall C$ if: $\forall ik \in L, \forall \pi_{ik} \in \Pi_{ik}, \forall m<k, \exists \pi_{im} \in \Pi_{im}$ s.t. $\{((\theta, a) \in \Theta \times A | \tau_{ik}(\theta, a_{<k}) \in \pi_{ik}\} \subseteq \{((\theta, a) \in \Theta \times A | \tau_{im}(\theta, a_{<m}) \in \pi_{im}\}$.

Let $\mathcal{F}$ denote the set of finite approximations with perfect recall.

Fact. For any information approximation $\Pi^0$, there is a finer approximation $\Pi$ such that $(\Pi, C)$ satisfies perfect recall for every $C$, and so $(\Pi, C) \in \mathcal{F}$ $\forall C$. 

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Strategy profiles for finite approximations

Here let \((\Pi, C) \in \mathcal{F}\) be a given finite approximation of the game and so \(S_{ik} \subseteq \Pi_{ik} \times C_{i(<k)}\) is the finite set of types of player \(ik\).

A strategy profile for the finite approximation \((\Pi, C)\) is any \(\sigma = (\sigma_{ik})_{ik \in L}\) s.t. each \(\sigma_{ik} : S_{ik} \to \Delta(C_{ik})\). So \(\sigma_{ik}(c_{ik}|s_{ik}) \geq 0, \forall c_{ik} \in C_{ik}, \sum_{\gamma \in C_{-ik}} \sigma_{ik}(\gamma|s_{ik}) = 1, \forall s_{ik} \in S_{ik}\).

Let \([t_{ik}]\) denote the element of \(\Pi_{ik}\) containing \(t_{ik} \in T_{ik}\).

\(\forall mble Z \subseteq \Theta\) and \(\forall c \in C\), let \(P(Z, c|\sigma) = \int_{\theta \in Z} (\prod_{ik \in L} \sigma_{ik}(c_{ik}[\tau_{ik}(\theta,c_{<k}],c_{i(<k)}))p(d\theta)\).

For any measurable \(R_{ik} \subseteq T_{ik}\), let \(P(R_{ik}|\sigma) = \sum_{c \in C} P(\{\theta \in \Theta| \tau_{ik}(\theta,c_{<k}) \in R_{ik}\},c|\sigma)\).

A totally mixed strategy profile \(\sigma\) for \((\Pi, C)\) has \(\sigma_{ik}(c_{ik}|s_{ik}) > 0, \forall c_{ik} \in C_{ik}, \forall s_{ik} \in S_{ik}\).

Any totally mixed \(\sigma\) for \((\Pi, C)\) yields \(P(R_{ik}|\sigma) > 0, \forall R_{ik} \in \mathcal{Q}_{ik}(C), \forall ik \in L\).

With \(P(R_{ik}|\sigma) > 0\), let \(P(Z, c|R_{ik}, \sigma) = P(Z \cap \{\theta|\tau_{ik}(\theta,c_{<k}) \in R_{ik}\},c|\sigma) / P(R_{ik}|\sigma)\).

We can extend this probability function to define \(P(Y|R_{ik}, \sigma) = \sum_{c \in C} P(Y(c),c|R_{ik}, \sigma)\) for any \(Y \subseteq \Theta \times A\) such that, for each \(c \in C\), \(Y(c) = \{\theta| (\theta,c) \in Y\}\) is a measurable subset of \(\Theta\).

For all such \(Y\) and \(\forall c_{ik} \in C_{ik}\), define \(P(Y|R_{ik}, c_{ik}, \sigma) = P(Y|R_{ik}, (c_{ik}, \sigma_{-ik}))\).

Note. Changing the action of \(i\) at \(k\) does not change the probability of \(i\)'s types at \(k\), so \(P(R_{ik}|c_{ik}, \sigma_{-ik}) = P(R_{ik}|\sigma) > 0\).
Approximate sequential equilibria
Here let $(\Pi, C) \in \mathcal{F}$ be a given finite approximation of the game, and let $\sigma$ be a totally mixed strategy profile for $(\Pi, C)$. With the probability function $P$ defined above, we can define
\[ V_i(\sigma|R_{ik}) = \sum_{c \in C} \int_{\theta \in \Theta} v_i(\theta, c) \ P(d\theta, c|R_{ik}, \sigma), \ \forall R_{ik} \in \mathcal{C}_{ik}(C). \]

For any $ik \in L$ and any $c_{ik} \in C_{ik}$, let $(c_{ik}, \sigma_{-ik})$ denote the strategy profile that differs from $\sigma$ only in that $i$ chooses action $c_{ik}$ at $k$ with probability 1. Changing the action of $i$ at $k$ does not change the probability of $i$'s types at $k$, so
\[ P(R_{ik}|c_{ik}, \sigma_{-ik}) = P(R_{ik}|\sigma) > 0, \ \forall R_{ik} \in \mathcal{C}_{ik}(C). \]
So we can similarly define $V_i(c_{ik}, \sigma_{-ik} | R_{ik})$ to be $i$'s payoff from choosing action $c_{ik}$ at period $k$, given the observable event $R_{ik}$, if others apply the $\sigma$ strategies.

For $\varepsilon > 0$, $\sigma$ is an $\varepsilon$-approximate sequential equilibrium for the finite approximation $(\Pi, C)$ iff $\sigma$ is a totally mixed strategy for $(\Pi, C)$ and
\[ V_i(c_{ik}, \sigma_{-ik}|s_{ik}) \leq V_i(\sigma|s_{ik}) + \varepsilon, \ \forall ik \in L, \ \forall s_{ik} \in S_{ik} \cap \mathcal{C}_{ik}(C), \ \forall c_{ik} \in C_{ik}. \]

Facts. Any finite approximation has an $\varepsilon$-approximate sequential equilibrium. In finite games with perfect recall, a strategy profile is part of a sequential equilibrium if and only if it is the limit as $\varepsilon \to 0$ of a sequence of $\varepsilon$-approximate sequential equilibria.
Assessments
Let \( Y \) denote the set of all outcome events \( Y \subseteq \Theta \times A \) such that \( \{\theta| (\theta,a) \in Y\} \) is a measurable subset of \( \Theta \), \( \forall \ a \in A \).

For any action approximation, \( C \), let \( W(C) = \mathcal{Q}(C) \cup (\cup_{ik \in L} \mathcal{Q}_{ik}(C) \times C_{ik}) \).

Let \( W = \mathcal{Q} \cup (\cup_{ik \in L} \mathcal{Q}_{ik} \times A_{ik}) \) (the union of all such \( W(C) \)).

An assessment is a vector \( \mu \) of conditional probabilities \( \mu(Y|W) \in [0,1] \ \forall \ Y \in Y, \forall \ W \in W \).

So \( \mu \) is in the compact (product topology) set \([0,1]^{Y \times W} \),
i.e., \( \mu(Y|Q_{ik}) \in [0,1] \) and \( \mu(Y|Q_{ik},a_{ik}) \in [0,1] \), \( \forall ik \in L, \forall Y \in Y, \forall Q_{ik} \in \mathcal{Q}_{ik}, \forall a_{ik} \in A_{ik} \).

Basic sequential equilibria
An assessment \( \mu \) is a basic sequential equilibrium iff: for every \( \varepsilon > 0 \), for every finite subset \( \Phi \) of \( Y \times W \), for every information approximation \( \Pi^0 \), there is a finer information approximation \( \Pi \) such that for every action approximation \( C^0 \), there is an action approximation \( C \supseteq C^0 \) such that \( \Phi \subseteq Y \times W(C) \), \( (\Pi,C) \in \mathcal{F} \) and \( (\Pi,C) \) has some \( \varepsilon \)-approximate sequential equilibrium \( \sigma \) such that
\[ |\mu(Y|W) - P(Y|W,\sigma)| \leq \varepsilon, \forall (Y,W) \in \Phi. \]

Theorem. The set of basic sequential equilibria is nonempty and is a closed subset of \([0,1]^{Y \times W} \) in the product topology.

The proof is based on Tychonoff's Theorem on compactness of products of compact sets. Kelley, 1955, p143.
Defining basic sequential equilibria with nets
A partially ordered set is *directed* if for any two members there is a member at least as large as both. A *net of real numbers* is a collection of real numbers indexed by a directed set \((D, \geq)\). A net of real numbers \(\{x_\delta\}_{\delta \in D}\) converges to the limit \(x^0\) if for every \(\varepsilon > 0\) there is an index \(\delta_\varepsilon\) s.t. \(x_\delta\) is within \(\varepsilon\) of \(x^0\) whenever \(\delta \geq \delta_\varepsilon\). Write \(\lim_{\delta \in D} x_\delta = x^0\). Also, define \(\underline{\lim}_{\delta \in D} x_\delta = \lim_{\delta \in D} (\inf_{\delta \geq \delta'} x_{\delta'})\).

The set of action approximations, partially ordered by set inclusion, and the set of information approximations, partially ordered by the fineness of the type-space partitions, are directed sets.

Then \(\mu\) is a basic sequential equilibrium iff for every finite subset \(\Phi\) of \(Y \times W\), there is a \(\sigma[\cdot]\) mapping such that

\[
\lim_{\varepsilon \to 0^+} \underline{\lim}_\Pi \underline{\lim}_C: (\Pi, C) \in \mathcal{F} \max_{(Y, W) \in \Phi} |\mu(Y|W) - P(Y|W, \sigma[\varepsilon, \Pi, C])| = 0,
\]

where each \(\sigma[\varepsilon, \Pi, C]\) is some \(\varepsilon\)-approximate sequential equilibrium of the finite approximation \((\Pi, C)\).

Note. All of the finitely many conditioning events determined by \(\Phi\) are given positive probability by \(\sigma[\varepsilon, \Pi, C]\) for all large enough \(C\).
Elementary properties of basic sequential equilibria
For any dated player ik and any observable event $Q_{ik} \in \mathcal{Q}_{ik}$, let
$I(Q_{ik}) = \{ (\theta,a) \in \Theta \times A \mid \tau_{ik}(\theta,a_{<k}) \in Q_{ik} \}.$

Let $\mu$ be a basic sequential equilibrium.
Then $\mu$ has the following general properties, for any outcome events $Y$ and $Z$ and any observable events $Q_{ik}$ and $Q_{jm}$:
\[
\mu(Y|Q_{ik}) \in [0,1], \quad \mu(\Theta \times A | Q_{ik}) = 1, \quad \mu(\emptyset | Q_{ik}) = 0 \quad (probabilities);
\]
if $Y \cap Z = \emptyset$ then $\mu(Y \cup Z | Q_{ik}) = \mu(Y | Q_{ik}) + \mu(Z | Q_{ik})$ (finite additivity);
$\mu(Y|Q_{ik}) = \mu(Y \cap I(Q_{ik})|Q_{ik})$ (conditional support);
$\mu(Y \cap I(Q_{jm})|Q_{ik}) \mu(I(Q_{ik})|Q_{jm}) = \mu(Y \cap I(Q_{ik})|Q_{jm}) \mu(I(Q_{jm})|Q_{ik})$ (Bayes consistency).

Bayes consistency implies that $\mu(Y|T_{ik}) = \mu(Y|T_{jm})$, for all $ik$ and $jm$ in $L$.
So the unconditional distribution on outcomes $\Theta \times A$ for a basic sequential equilibrium can be defined by $\mu(Y) = \mu(Y|T_{ik})$, $\forall Y \subseteq \Theta \times A$, $\forall ik \in L$.
The unconditional marginal distribution of $\mu$ on $\Theta$ is the given prior $p$:
$\mu(Z \times A) = p(Z)$, for any measurable $Z \subseteq \Theta$.

Fact (sequential rationality). If $\mu$ is a basic sequential equilibrium then, $\forall ik \in L$
$\forall Q_{ik} \in \mathcal{Q}_{ik}$, $\forall a'_{ik} \in A_{ik}$, $\int v_i(\theta,a) \mu(d\theta,da|Q_{ik}) \geq \int v_i(\theta,a) \mu(d\theta,da|Q_{ik},a'_{ik}).$
The left-hand side integral is player $i$’s equilibrium payoff conditional on $Q_{ik}$. The right-hand side integral is $i$’s payoff from deviating at date $k$ to $a'_{ik}$ conditional on $Q_{ik}$.
Finite additivity of solutions

*Example:* Consider a game where the winner is the player who chooses the smallest strictly positive number. In any finite approximation, let each player choose the smallest strictly positive number that is available to him.

In the limit, we get $\mu(0 \leq a_1 \leq x) = 1$, $\forall x > 0$, but $\mu(a_1 = 0) = 0$.

The probability measure $\mu$ is not countably additive, only finitely additive, and this finite additivity lets us represent an infinitesimal positive action $a_1$.

Expected utilities are well-defined with finite additivity, as utility is bounded.

Suppose the players are $\{1,2\}$, and player 2 observes $a_1$ before choosing $a_2$. For any possible value of $a_1$, 2 would always choose $a_2 < a_1$ after observing this $a_1$ in any sufficiently large finite approximation. So in the limit, we get:

$\forall x > 0$, $\mu(a_2 < a_1 | a_1 = x) = 1$ and $\mu(0 < a_1 \leq x) = 1$.

But multiple equilibria can have any prior $P(2 \text{ wins}) = \mu(a_2 < a_1)$ in $\{0,1\}$. By considering a subnet of finite approximations in which 1 always has smaller actions than 2, we can get an equilibrium with $\mu(a_2 < a_1) = 0$ even though $\mu(a_2 < a_1 | a_1 = x) = 1$ $\forall x > 0$ and $\mu(a_1 > 0) = 1$.

In this subnet, 1 always chooses $a_1$ where 2's strategy has not converged, and so the outcome cannot be derived by backward induction from 2's limiting strategy.

Strategic actions avoid such problems, as would considering subnets where later players' actions grow faster than earlier players' (fast inner limits for later $C_{\bullet k}$).
[Games with perfect information]
If $\Theta=\{\theta_0\}$, then a dynamic multi-stage game has perfect information iff:
\[ \forall k \in \{1,\ldots,K\}, |A_{ik}|>1 \text{ for at most one player } i \in N, \text{ and } \]
\[ \forall ik \in L, \tau_{ik}: \{\theta_0\} \times A_{<k} \rightarrow T_{ik} \text{ is one to one}. \]
(No two players make choices at the same date, and all players observe the past history of play.)

Theorem. Every dynamic multi-stage game of perfect information with continuous payoffs, compact Hausdorff action sets and one state of nature, i.e., $\Theta=\{\theta_0\}$, has a basic sequential equilibrium, $\mu$, in which all prior probabilities are 0 or 1. That is,
\[ \mu(\{\theta_0\} \times B) \in \{0,1\}, \forall B \subseteq A. \]

Proof: Fix $\delta \in (0,1)$. For every $\varepsilon>0$ and every information approximation $\Pi^0$, by compactness and continuity, there is a finer information approximation such that the last player has, for each element of his partition, a single action that is $\varepsilon$-optimal against all histories leading to that partition element. With this strategy fixed for the last player we may repeat the argument for the second last player after possibly further refining his information partition, etc. Hence, for each $\varepsilon>0$ there is an information approximation $\Pi$ that is finer than $\Pi^0$ and is such that $\forall C$ large enough the game $(\Pi,C)$ admits an $\varepsilon$-approximate sequential equilibrium in which some outcome receives probability at least $1-\delta$, so that the probability of any $B \subseteq A$ is in $[0,\delta] \cup [1-\delta,1]$. Letting $\delta \rightarrow 0$ yields the result because the set of basic sequential equilibria is closed in the product topology.
Strategic entanglement (Milgrom-Weber 1985)

**Example:** $\theta$ is uniform $[0,1]$, observed by both players 1 and 2, who then simultaneously play a battle-of-sexes game where $A_1=A_2=\{1,2\}$, $v_i(\theta,a_1,a_2) = 2$ if $a_1=a_2=i$, $v_i = 1$ if $a_1=a_2 \neq i$, and $v_i = 0$ if $a_1 \neq a_2$.

Consider finite-approximation equilibria such that, for a large integer $m$, they do $a_1=a_2=1$ if the $m$'th digit of $\theta$ is odd, $a_1=a_2=2$ if the $m$'th digit of $\theta$ is even.

As $m \to \infty$, these equilibria converge to a limit where the players randomize between actions $(1,1)$ and $(2,2)$, each with probability $1/2$, independently of $\theta$.

In the limit, the players' actions are not independent given $\theta$ in any positive interval. They are correlated by commonly observed infinitesimal details of $\theta$.

Given $\theta$ in any positive interval, player 1's expected payoff is 1.5 in eqm, but 1's conditional payoff from deviating to $a_1=1$ is $0.5(2)+0.5(0) = 1$, and 1's conditional payoff from deviating to $a_1=2$ is $0.5(0)+0.5(1) = 0.5$.

Thus, the sequential-rationality inequalities can be strict for all actions. To tighten the sequential-rationality lower bounds for conditional expected payoffs in equilibrium, we could consider also deviations of the form: "deviate to action $c_{ik}$ when the equilibrium strategy would select an action in the set $B_{ik}$" for any $c_{ik} \in A_{ik}$ and any $B_{ik} \subseteq A_{ik}$.

In the next example, strategic entanglement is unavoidable, occurring in the only basic sequential equilibrium.
[(Unavoidable) **Strategic entanglement**] (Harris-Reny-Robson 1995)

**Example:** Date 1: Player 1 chooses $a_1$ from $[-1,1]$, player 2 chooses from \{L,R\}.
Date 2: Players 3 and 4 observe the date 1 choices and each choose from \{L,R\}.

For $i=3,4$, player $i$’s payoff is $-a_1$ if $i$ chooses L and $a_1$ if $i$ chooses R.

Player 2’s payoff depends on whether she matches 3’s choice.
If 2 chooses L then she gets 1 if player 3 chooses L but -1 if 3 chooses R; and
If 2 chooses R then she gets 2 if player 3 chooses R but -2 if 3 chooses L.

Player 1’s payoff is the sum of three terms:
(First term) If 2 and 3 match he gets $-|a_1|$, if they mismatch he gets $|a_1|$;
plus (second term) if 3 and 4 match he gets 0, if they mismatch he gets -10;
plus (third term) he gets $-|a_1|^2$.

Approximations in which 1’s action set is \{-1,\ldots,-2/m,-1/m,1/m,2/m,\ldots,1\} have a unique
subgame perfect (hence sequential) equilibrium in which player 1 chooses $\pm 1/m$ with
probability $\frac{1}{2}$ each, player 2 chooses L and R each with probability $\frac{1}{2}$, and players $i=3,4$
both choose L if $a_1=-1/m$ and both choose R if $a_1=1/m$.
Player 3’s and player 4’s strategies are entangled in the limit.

The limit of every approximation produces (the same) strategic entanglement.
[(Spurious) **Strategic entanglement**] (Harris, Stinchcombe, Zame 2000)

**Example:** $\theta$ is uniform $[0,1]$, payoff irrelevant, observed by both players 1 and 2, who then simultaneously play the following 2x2 game.

\[
\begin{array}{c|cc}
 & L & R \\
\hline
L & 1,1 & 3,2 \\
R & 2,3 & 0,0 \\
\end{array}
\]

Consider finite approximations in which, for large integers $m$, each player observes the first $m-1$ digits of the ternary expansion of $\theta$. Additionally, player 1 observes whether the $m$’th digit is 1 or not and player 2 observes whether the $m$’th digit is 2 or not.

Consider equilibria in which each player $i$ chooses $R$ if the $m$’th digit of the ternary expansion of $\theta$ is $i$ and chooses $L$ otherwise. As $m \to \infty$, these converge to a correlated equilibrium of the 2x2 game where each cell but (R,R) obtains with probability $1/3$, independently of $\theta$. Not in the convex hull of Nash equilibria!

The type of strategic entanglement generated here is impossible to generate in some approximations—it depends on the fine details—unlike the entanglement in the previous two examples.
Problems of spurious concealment in approximating games

Example. The state \( \theta \) is a 0-or-1 random variable, which is observed by player 1 as his type \( t_1 \). Then player 1 chooses a number \( a_1 \) in \([0,1]\), which is subsequently observed by players 2 and 3.

Consider a finite approximation in which 1 observes \( t_1 \), and then can choose any \( m \)-digit decimal number in \([0,1]\).

Suppose that player 1 wants to share information with 2 but conceal it from 3.

Such concealed signaling would be possible in a finite approximation where 2 can observe 1's action exactly while 3 can observe only its first \( m-1 \) digits.
But in the real game, any message that 1 sends to 2 is also observed by 3, and it should not be possible for 1 and 2 to tunnel information past 3.

The concealed-signaling equilibrium here depends on the fine details of the approximation.

Possible solution: Require that type sets and action sets which are "the same" must have the same finite approximations. (How to recognize sameness?)
Example: Three players choose simultaneously from $[0,1]$. Player 1 wishes to choose the smallest positive number. Players 2 and 3 wish to match player 1, receiving 1 if they do but 0 otherwise.

It should not be possible for 2, but not 3, to match 1.

But consider finite approximations in which the smallest positive action available to player 1 is available to player 2, but not to player 3.

The only equilibria have 1 and 2 choosing that special common action and player 3 choosing any (other) action. Hence, there are basic sequential equilibria in which 2 matches 1 but 3 does not.

The exclusive-coordination equilibrium here depends on the fine details of the approximation.

Possible solution: Require that type sets and action sets which are "the same" must have the same finite approximations. (How to recognize sameness?)
Limitations of step-strategy approximations

**Example:** θ is uniform [0, 1], player 1 observes θ, chooses $a_1 \in [0, 1]$, $v_1(\theta, a_1) = 1$ if $a_1 = \theta$, $v_1(\theta, a_1) = 0$ if $a_1 \neq \theta$.

In any finite approximation, player 1's finite action set cannot allow any positive probability of $c_1$ matching θ exactly, and so 1's expected payoff must be 0.

Player 1 would like to use the strategy "choose $a_1 = \theta$," but this strategy can be only approximated by step functions when 1 has finitely many feasible actions. Step functions close to this strategy yield very different expected payoffs because the utility function is discontinuous.

If we gave player 1 an action that simply applied this strategy, he would use it!

(It may be hard to choose any real number exactly, but easy to say "I choose θ.")

Thus, adding a *strategic action* that implements a strategy which is feasible in the limit game can significantly change our sequential equilibria.

This problem follows from our principle of finitely approximating information and actions separately, which is needed to prevent spurious signaling.

**Example (Akerlof):** θ uniform[0, 1], 1 observes $\tau_1 = \theta$, 1 chooses $a_1 \in [0, 1.5]$, 2 observes $\tau_2 = a_1$, 2 chooses $a_2 \in \{0, 1\}$, $v_1(\theta, a_1, 1) = a_1 - \theta$, $v_2(\theta, a_1, 1) = 1.5\theta - a_1$, $v_i(\theta, a_1, 0) = 0$. Given any finite set of strategies for 1, we could always add some strategy $b_1$ such that $\theta < b_1(\theta) < 1.5\theta \ \forall \theta$, and $b_1(\theta)$ has probability 1 of being in a range that has probability 0 under the other strategies in the given set.
Using strategic actions to solve problems of step-strategy approximations
One way to avoid the problems of step-strategies is to permit the use of strategic actions. This must be done with care to avoid the problem of spurious signaling.
Algebras (closed under finite $\cap$ and complements) of measurable subsets are specified also for each $A_{ik}$, including all one-point sets as measurable.
If $\Theta$ is a finite set, then all subsets of $A_{ik}$ can be assumed measurable.
For any msble $g: \Theta \rightarrow A$, all $\tau_{ik}(\theta,g(\theta)_{<k})$ and $v_i(\theta,g(\theta))$ are assumed msble in $\theta$.
A strategic action for $ik \in L$ is a measurable function, $a_{ik}*:T_{ik} \rightarrow A_{ik}$.
Let $A_{ik}^*$ denote ik’s set of strategic actions.
The set of intrinsic actions $A_{ik}$ is a subset of $A_{ik}^*$ because for each $a_{ik} \in A_{ik}$, $A_{ik}^*$ contains the constant strategic action taking the value $a_{ik}$ on $T_{ik}$.
For every $a^* \in A^*$ and every $\theta \in \Theta$, let $\alpha(\theta,a^*)$ denote the action profile $a \in A$ that is determined when the state is $\theta$ and each player plays according to the strategic action profile $a^*$. Note that for each $a^* \in A^*$, $\alpha(\theta,a^*)$ is measurable in $\theta$.
To ensure perfect recall, define new type spaces and type maps:
$T_{ik}^* = T_{ik} \times (\times_{h<k} A_{ih}^*)$, with measurable subsets whose slices contained in $T_{ik}$ defined by any $a_{<k}^*$ are measurable, and $\tau_{ik}^*(\theta,a_{<k}^*)=(\tau_{ik}(\theta,\alpha(\theta,a^*)_{<k}), \times_{h<k} a_{ih}^*)$.
Extend each $v_i(\theta,a)$ from $\Theta \times A$ to $\Theta \times A^*$ by defining $v_i(\theta,a^*)= v_i(\theta,\alpha(\theta,a^*))$.
Note that for every $ik \in L$ and every $a^* \in A^*$, $\tau_{ik}^*(\theta,a_{<k}^*)$ and $v_i(\theta,a^*)$ are measurable functions of $\theta$. 
This defines a dynamic multi-stage game, $\Gamma^*$, with perfect recall and we may therefore employ all of the notation and definitions we previously developed.

$\mathcal{Q}_{ik}^*$ is the set of observable events for player $i$ at period $k$ that can have positive probability in $\Gamma^*$. $\mathcal{Q}^*$ is the (disjoint) union of these $\mathcal{Q}_{ik}^*$ over all $ik$.

Approximations of $\Gamma^*$ are defined, as before, by $(\Pi, C)$, where $C$ is any finite set of (strategic) action profiles in $A^*$ and $\Pi$ is any information approximation of all the $T_{ik}^*$.

**Perfect recall in finite approximations of $\Gamma^*$**

In a finite approximation of $\Gamma^*$, perfect recall means that a player $ik$ remembers, looking back to a previous date $m<k$, the partition element containing his information-type at that date and the strategic action chosen there.

If the partition element contains more than one of his information types and his strategic action there is not constant across the projection of those types onto $T_{im}$, then he will not recall the action in $A_{im}$ actually taken at the previous date.

Thus, there is perfect recall with respect to strategic actions but not with respect to intrinsic actions.

Of course, if the strategic action chosen is a constant function, i.e., equivalent to an intrinsic action, then the action too is recalled.
Robustness to approximations

The problems of spurious strategic entanglement, spurious concealment, and spuriously exclusive coordination arise when limits of approximate equilibria depend on the fine details of the approximation. One might then insist that equilibria be robust to \textit{all} approximations. But this is not compatible with existence (e.g., two players each wish to choose the smallest strictly positive number).

An alternative is to seek “as much robustness as possible” subject to existence.

For \( \sigma \) completely mixed in \((\Pi,C)\), let \( m(\sigma) \in [0,1]^{y \times w} \) be the assessment whose probabilities agree with \( P(Y|W,\sigma) \) when \((Y,W)\in y \times w(C)\), and are zero otherwise.

\[ \text{Essential sequential equilibria} \]

The set of \textit{essential sequential equilibria} is the smallest closed set of assessments containing every closed set of assessments \( M \) that is minimal with respect to the following property: for every open set \( U \) containing \( M \), there is a \( \sigma[\cdot] \) mapping such that

\[ \lim_{\epsilon \to 0^+} \lim_{\Pi} \lim_{C:(\Pi,C) \in \mathcal{F}} \mathbf{I}_U(m(\sigma[\epsilon,\Pi,C])) = 1, \]

where \( \mathbf{I}_U(\cdot) \) is the indicator function for \( U \), and each \( \sigma[\epsilon,\Pi,C] \) is some \( \epsilon \)-approximate sequential equilibrium of the finite approximation \((\Pi,C)\).

\textit{Theorem}. The set of essential sequential equilibria is nonempty and is a closed subset of the set of basic sequential equilibria.

(By Zorn’s lemma and Tychonoff’s Theorem \( \exists \) such a minimal set \( M \) that is nonempty.\textsuperscript{24}
References:
Christopher J. Harris, Maxwell B. Stinchcombe, and William R. Zame, "The Finitistic Theory of Infinite Games," UTexas.edu working paper.