A SHORT INTRODUCTION TO THE PRIME NUMBER THEOREM

IRVING DAI

Abstract. We briefly discuss the history of the prime number theorem and summarize Newman’s 1980 proof using complex analysis. I have attempted to convey the flavor of the proof without deviating too much from heuristic arguments; all details are left as exercises to the reader.

“There are two facts about the distribution of prime numbers of which I hope to convince you so overwhelmingly that they will be permanently engraved in your hearts. The first is that, despite their simple definition and role as the building blocks of the natural numbers, the prime numbers grow like weeds among the natural numbers, seeming to obey no other law than that of chance, and nobody can predict where the next one will sprout. The second fact is even more astonishing, for it states just the opposite: that the prime numbers exhibit stunning regularity, that there are laws governing their behavior, and that they obey these laws with almost military precision.” - D. Zagier

1. Introduction and History

Understanding the distribution of the primes has long remained one of the most enduring and difficult problems in number theory. In 1797, Legendre proposed perhaps the most well-known theorem to this effect:

Theorem 1.1 (Prime Number Theorem). Let \( \pi(x) \) denote the number of primes less than or equal to \( x \). Then \( \pi(x) \sim x/\log x \); i.e.,

\[
\lim_{x \to \infty} \pi(x) \left( \frac{\log x}{x} \right) \to 1.
\]

Legendre was anticipated by Gauss, who calculated (by hand) a table of all primes up to three million. Significant progress towards proving the prime number theorem was made in 1848 by Chebyshev, who showed that if the above limit exists at all then it must equal one. Full proofs were given independently by Hadamard and Vallée-Poussin in 1896; these were remarkable not only for the strength of the result but also due to the groundbreaking use of techniques from complex analysis introduced by Riemann half a century before. We briefly outline the main ideas of a simplified proof given by Newman in 1980 and conclude with some historical facts and further developments.

2. The Riemann Zeta Function

The first steps towards a proof of the prime number theorem were taken by Riemann in 1859, although many of his ideas had been previously anticipated by Chebyshev and Euler. As a formal series in \( s \), consider the Riemann zeta function

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.
\]

It is easily seen that \( \zeta \) has the equivalent expression

\[
\zeta(s) = \prod_{p \text{ prime}} \left( 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots \right) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}
\]

by unique prime factorization. This tentative connection between \( \zeta \) and the prime numbers was first observed by Euler, and indeed the proof of the prime number theorem involves several deep properties of the zeta function. Among these is the following identity involving the logarithmic derivative: letting

\[
\Lambda(n) = \begin{cases} 
\log p & \text{if } n = p^k \text{ for some prime } p \\
0 & \text{else}
\end{cases}
\]

be the von Mangoldt function, straightforward computation using the product form of \( \zeta \) yields

\[
\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.
\]
Such series (i.e., sums of the form $\sum f(n)/n^s$) are called **Dirichlet series**, and have many important properties and applications to number theory.

We now present the first insight into the proof of the prime number theorem and show how it is related to $\Lambda$ (and hence $\zeta$). Define

$$\psi(x) = \sum_{n \leq x} \Lambda(n).$$

Observing that $\psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{p \leq x \text{ prime}} (\log p)\lceil \log_p x \rceil \approx \pi(x) \log x$, we have

**Theorem 2.1.** $\pi(x) \sim x/\log x$ if and only if $\psi(x) \sim x$.

This relates the distribution of the primes to the coefficients of the Dirichlet expansion of $\zeta'/\zeta$.

3. **Analytic Elements**

Riemann’s essential insight was to consider Dirichlet series not just as formal expressions, but as actual analytic functions on the domain $\text{Re}(s) > 1$. By various theorems of complex analysis, the above manipulations hold as *bona fide* identities of functions. We may thus cast these equalities in analytic form; in particular, we have

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -s \int_{t=1}^{\infty} \psi(t)t^{-s-1}dt.$$

This follows from breaking up $[1, \infty)$ into pieces $[n, n+1)$ and observing that $\psi$ is constant on each such interval. The expression on the right is (essentially) called the **Mellin transform** of $\psi$ and is valid in the domain $\text{Re}(s) > 1$. An important breakthrough was the use of theorems about the Mellin (and related) transforms to establish properties of $\psi$. Observe that we have (being rather liberal with convergences)

$$-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} = s \int_{t=1}^{\infty} \psi(t)t^{-s-1}dt - \frac{1}{s-1} = s \int_{t=1}^{\infty} \psi(t)t^{-s-1}dt - \int_{t=1}^{\infty} t^{-s}dt = s \int_{t=1}^{\infty} \frac{\psi(t) - t/s}{t^{s+1}}dt$$

for $\text{Re}(s) > 1$. This seemingly uninspired equality assumes remarkable importance if we try to extend it to $s = 1$, for the right-hand side then becomes

$$\int_{t=1}^{\infty} \frac{\psi(t) - t}{t^2}dt$$

and it is not hard to see that this integral converges if and only if $\psi(x) \sim x$.

How might we hope to prove convergence of the above integral? The essential point is to observe that the left-hand side of the equality (1) has an analytic extension to a neighborhood of $\text{Re}(s) = 1$. This result rests on two main properties of $\zeta$. First, $\zeta$ has a meromorphic extension to the entire plane with a single pole of residue of 1 at $s = 1$. Second, $\zeta$ has no zeros in the half-plane $\text{Re}(s) > 1$, and (more importantly) no zeros on the line $\text{Re}(s) = 1$. This latter fact is not obvious and constitutes an essential part of the proof of the prime number theorem. The function $-\zeta'/\zeta$ thus has a meromorphic extension to a neighborhood of $\text{Re}(s) = 1$ with a simple pole of residue of 1 at $s = 1$.

Knowing that the left-hand side of (1) has an analytic extension to a neighborhood of $\text{Re}(s) = 1$, it thus seems reasonable to hope that the right-hand side will converge for $s = 1$. Theorems to this effect are called **Tauberian theorems** and are of great importance in analysis. In particular,

**Theorem 3.1** (Analytic Theorem). Let $f(t)$ be a bounded locally integrable function and suppose that its Laplace transform

$$g(s) = \int_{0}^{\infty} f(t)e^{-st}dt$$

is defined for $\text{Re}(s) > 0$ and extends holomorphically to $\text{Re}(s) = 0$. Then the integral

$$\int_{0}^{\infty} f(t)dt$$

exists and equals $g(0)$.

Although phrased in the language of Laplace transforms (rather than Mellin transforms), with some translation this yields the desired convergence.
4. Historical Remarks

The proof that we have sketched above is not the original proof given by Hadamard and Vallée-Poussin, but is similar in spirit. The analytic properties of $\zeta$ were central to the argument, and in particular the absence of zeros on the line $\text{Re}(s) = 1$ was of major importance. This has given rise to the popular assertion that a “one-line proof” of the prime number theorem consists of the fact that $\zeta$ has no such zeros. While this is rather unconvincing, it is certainly true that the zeros of $\zeta$ are intimately tied up with the distribution of the primes. Indeed, the most famous conjecture to this effect

**Conjecture 4.1** (Riemann Hypothesis). The nontrivial zeros of $\zeta$ all lie on the line $\text{Re}(s) = 1/2$.

is equivalent to the following strengthened version of the prime number theorem:

**Conjecture 4.2.** Let

$$Li(x) = \int_2^x \frac{1}{\log t} dt.$$  

Then

$$\pi(x) = Li(x) + O(\sqrt{x} \log x).$$

This formulation of the prime number theorem not only gives the asymptotic behavior of $\pi(x)$, but also provides a tight error bound.

Over the years, a number of alternate proofs of the prime number theorem have been proposed. Perhaps most historically interesting among these is the 1948 proof of Selberg and/or Erdös. There remains a bitter priority dispute over who deserves credit, but mathematically the proof was remarkable because no complex analysis was used. This laid to rest certain philosophical assertions popularized by Hardy that any proof of the prime number theorem would have to resort to “non-elementary” methods (although the Selberg/Erdös proof was certainly quite complicated).

We close by mentioning similar results in number theory also established through analysis. These include the celebrated Dirichlet theorem:

**Theorem 4.3** (Dirichlet’s Theorem). Let $a$ and $m$ be coprime. Then there exist infinitely many prime numbers $p$ for which $p = a \mod m$.

The prime number theorem and Dirichlet’s theorem comprise two of the most famous results in the field of analytic number theory and marked the beginning of the use of analysis in solving number-theoretic problems. Recent theorems to this effect include

**Theorem 4.4** (Green-Tao Theorem). The prime numbers contain arithmetic progressions of arbitrary length.

which was (amazingly) proven using techniques from ergodic theory.

5. References


