Automorphy lifting for residually reducible $l$-adic Galois representations

Jack A. Thorne

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Abstract

We prove automorphy lifting theorems for residually reducible Galois representations in the setting of unitary groups. Our methods are inspired by those of Skinner-Wiles in the setting of $GL_2$.

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1 Introduction

In this paper we prove new automorphy lifting theorems for $l$-adic Galois representations over CM fields satisfying a self-duality hypothesis. The main novelty is that we can prove lifting results for Galois representations which are residually reducible. This paper can therefore be viewed as a sequel to [Tho12], where we treated the residually irreducible case. However, there are a number of serious new obstacles.

Let $F$ be an imaginary CM field with totally real subfield $F^+$, and let $c \in \text{Gal}(F/F^+)$ denote the non-trivial element. Let $\rho : G_F \to \text{GL}_n(\overline{\mathbb{Q}}_l)$ be a continuous irreducible representation. We suppose that $\rho$ is conjugate self-dual, in the sense that $\rho^c \equiv \rho^c \epsilon^{1-n}$, where $c \in \text{Gal}(F/F^+)$ denotes the non-trivial element and $\epsilon$ the $l$-adic cyclotomic character, and de Rham with distinct Hodge-Tate weights. The central problem for us is to show that $\rho$ is automorphic, in the sense of Theorem 2.1 below. Choosing a suitable finite extension $K$ of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$ with ring of integers $\mathcal{O}_K$ and an invariant $\mathcal{O}_K$-lattice inside $K^n$, we can view $\rho$ as a representation $G_F \to \text{GL}_n(\mathcal{O}_K)$. Then (writing $\lambda \subset \mathcal{O}_K$ for the unique maximal ideal) the reduced representation $\overline{\rho} = \rho \mod \lambda$ makes sense, and its semisimplification $\overline{\rho}^{ss}$ is independent of the choice of invariant lattice. Previous efforts have centered around the case where $\overline{\rho}$ is absolutely irreducible. In this case, assuming the existence of a lift of $\overline{\rho}$ which arises from automorphic forms, we obtain a map $\varphi : R \to T$, where $R$ is the universal deformation ring of $\overline{\rho}$ classifying deformations of a certain well-chosen type (for example, conjugate self-dual, de Rham with fixed Hodge-Tate weights), and $T$ is the quotient classifying deformations which arise from a suitable space of automorphic forms (for example, automorphic forms on a unitary group which have cohomology for a fixed local system). We can then proceed by trying to show that $\varphi$ is close to being an isomorphism.

If $\overline{\rho}$ is not absolutely irreducible, then problems arise. First, the universal deformation ring need not exist in general. For some choices of invariant lattice, one can arrive at a $\overline{\rho}$ with scalar centralizer. (This is the approach adopted in the work of Skinner-Wiles [SW99], where the authors choose $\overline{\rho}$ to take values in the group $B_2 \subset \text{GL}_2$ of upper-triangular matrices.) In this case, the universal deformation ring exists, but there need not exist a map $R \to T$. In general one can expect a map $R^{tr} \to T$, where $R^{tr}$ denotes the universal pseudodeformation ring of the pseudomulticharacter $\text{tr} \overline{\rho}$, but the ring $R^{tr}$ is difficult to control using Galois cohomology, a tool which is essential in other arguments.

In this paper we circumvent these problems in some cases by permitting $\overline{\rho}$ which are ‘reducible, but not too reducible’. In fact, we allow residual representations which are Schur, cf. Definition 3.2 below. (This property was first defined in [CHT08].) If $\overline{\rho}$ is Schur then the universal deformation ring $R$ exists, and is related to the universal pseudodeformation ring $R^{tr}$ in a simple way. This behavior is related to the existence of elliptic endoscopic groups of $U(n)$.

Having restricted our attention to this class of representations, we try to apply the Taylor-Wiles-Kisin argument. The relevant arguments in Galois cohomology require the residual representation to be absolutely irreducible (at the very least; current technology asks for it further to be adequate, in the sense of [Tho12]). In order to circumvent this difficulty, we follow the strategy of Skinner-Wiles [SW99], who have proved automorphy lifting theorems for $\text{GL}_2$, working with residually reducible representations over totally real fields. The basic idea is that by working with Hida families we can move from the residual representation $\overline{\rho}$ to an irreducible representation with coefficients in a one-dimensional quotient of the Iwasawa algebra. One can then try to apply the usual arguments to a localization of $R$ at the dimension one prime corresponding to such a representation. (This means we must restrict to representations $\rho$ which are not only de Rham but even ordinary at primes dividing $\ell$.)

There is one final hiccup. At a key point in the argument, we must show that the locus inside $\text{Spec } R$ of reducible Galois representations has large codimension. In contrast to the case of $\text{GL}_2$, when working with unitary Galois representations, there is no a priori reason for this to be the case; the endoscopic parameters can contribute irreducible components of $\text{Spec } R$ which have full dimension. (In the context in which we work, one expects a universal deformation ring allowing representations ordinary at $\ell$, of variable Hodge-Tate...
weights, to be equidimensional of dimension $1 + nd$, where $d = [F^+ : \mathbb{Q}]$.) For this reason, we must impose an additional hypothesis. For example, we can ask for $p$ to admit a place $v$ at which the associated Weil-Deligne representation of $\rho|_{G_{F_v}}$ corresponds under the local Langlands correspondence to a twist of the Steinberg representation. Since this is incompatible with $\rho$ being a direct sum of two representations of strictly smaller dimension, we can force the locus of reducible deformations to be small. Our main theorem, Theorem 7.1, is an automorphy lifting theorem which makes use of this assumption.

Let us briefly describe one possible application of our work. Part of the interest of automorphy lifting theorems in the residually reducible case is that it is often easier to verify the residual automorphy hypothesis. For example, Skinner-Wiles take the approach of showing that the constant term of a GL$_2$ Eisenstein series vanishes mod $l$, so one can apply the Deligne-Serre lemma to obtain a congruence with cusp form. A different approach can be taken with automorphic forms on unitary groups. For example, one can take an endoscopic lift from a product of smaller unitary groups and then apply a level-raising result (as in, for example, the work of Bellaïche-Graftieaux [BG06] or our paper [Tho]) to obtain a congruence with an automorphic representation which is stable. Since the automorphic representations we eventually consider are for other reasons assumed to be square integrable at a finite place, this approach seems particularly effective here. In the final section below we discuss a theorem which combines this idea with Serre’s conjecture for GL$_2$ over $\mathbb{Q}$ (now a theorem of Khare-Wintenberger and Kisin) to prove an automorphy result for irreducible three-dimensional Galois representations over a quadratic imaginary field, with no hypothesis of residual automorphy.

We now describe the organization of this paper. In §2, we recall the definition of a RAECSDC automorphic representation of GL$_n(A_F)$. In §3, we describe the basic objects in deformation theory with which we work. In particular, we make our first important observation, about the relation between the rings $R$ and $R^{tr}$ (denoted $R^{\text{inv}}_p$ and $P_S$ in the body of the paper). Namely, we show that when $\overline{p}$ is Schur, the natural map $R^{tr} \to R$ is a finite ring homomorphism. This generalizes the well-known fact (due to Carayol for GL$_n$) that if $\overline{p}$ is absolutely irreducible, then $R^{tr} \to R$ is surjective.

In §4, we define the relevant spaces of automorphic forms and recall some basic facts from Hida theory. We prove an ‘$R_p = T_p$’ theorem under some stringent hypotheses. Here $p$ denotes a dimension one prime of $R$, and $(\cdot)_p$ denotes localization and completion at that prime. §5 is devoted to giving some situations when these hypotheses can be expected to hold.

In §6, we show how to upgrade an ‘$R_p = T_p$’ theorem into information about the relation between $R$ and $T$. Since $R_p$ only knows about the irreducible components containing $p$, we need a way to move between different components of Spec $R$, to do this we use some input from commutative algebra, in the form of the notion of connectedness dimension of local rings. In §7 we give our main result, an automorphy lifting theorem using all of the ideas discussed in this introduction. Finally, in §8 we describe an application of our work to the Fontaine-Mazur conjecture for $U(3)$.

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Notation

If $F$ is a field of characteristic zero, we write $G_F$ for its absolute Galois group. If $F/F^+$ is a quadratic extension of such fields, we write $\delta_{F/F^+}$ for the non-trivial character of $\text{Gal}(F/F^+)$. We write $\epsilon_l : G_F \to \mathbb{Z}_l^\times$ for the $l$-adic cyclotomic character. If the prime $l$ is understood, we will write $\epsilon_l = \epsilon$.

We fix throughout this paper an algebraic closure $\overline{Q}_l$ of $Q_l$ for each prime $l$. If $F$ is a number field and $\chi$ is a character $A_F^\times / F^\times \to \mathbb{C}^\times$ of type $A_0$ (i.e. the restriction of $\chi$ to $(F \otimes \mathbb{R})^\times_0$ is given by $\prod_{\tau: F \to \mathbb{C}} x_\tau^{a_\tau}$ for some integers $a_\tau$), and $\iota$ is an isomorphism $Q_l \cong \mathbb{C}$, then we write $r_\iota(\chi) : G_F \to \overline{Q}_l^\times$ for the associated
$l$-adic character, given by the formula

$$
ℓ \left( r_ι(χ) \circ \text{Art}_F(x) \right) = \prod_{τ ∈ \text{Hom}(F, ℂ)} x_τ^{−a_τ} = χ(x) \prod_{τ ∈ \text{Hom}(F, ℂ)} x_τ^{−a_τ},
$$

where $\text{Art}_F$ is the global Artin map

$$
\text{Art}_F = \prod_v \text{Art}_{F_v} : A_F^x → G_F^{ab}.
$$

We normalize the local Artin maps $\text{Art}_{F_v}$ to take uniformizers to geometric Frobenii.

If $F$ is a number field and $v$ is a finite place of $F$ with completion $F_v$, we will write $O_{F_v}$ for its ring of integers, $k(v)$ for its residue field, and $q_v$ for the cardinality of $k(v)$. We will write $\text{Frob}_v$ for the geometric Frobenius element in $G_{F_v}/I_{F_v}$. Suppose that $ρ : G_{F_v} → GL_n(ℚ_l)$ is a continuous representation, and take an embedding $τ : F_v ↪ ℚ_l$. We write $HT_τ(ρ)$ for the multiset of integers whose elements are the integers $i$ such that $gr^i(ρ ⊗ τ, F)B_{dR}G_{F_v} ≠ 0$, with multiplicity $dim_{ℚ_l} gr^i(ρ ⊗ τ, F, B_{dR})Gr_{F_v}$. Here $B_{dR}$ denotes Fontaine’s ring of $p$-adic (or $l$-adic) periods, cf. [Ber04]. If $F$ is a number field and $ρ : G_F → GL_n(ℚ_l)$ is a continuous representation, and $τ : F ↪ ℚ_l$, we write $HT_τ(ρ)$ to mean $HT_τ(ρ|_{G_{F_v}})$, where $v$ is the place of $F$ induced by the embedding $τ$. Thus for the character $r_ι(χ)$ defined above, we have

$$HT_τ(r_ι(χ)) = \{−a_τ\},$$

and $HT_τ(ε) = \{−1\}$ for any $τ$.

If $π$ is an irreducible admissible representation of $GL_n(F_v)$ over $ℚ_l$ then we will use the notation $r_ι(π)$ introduced in [CHT05] §3.1 to denote the $l$-adic representation associated to $π$ under the local Langlands correspondence, when it exists. We write $ℤ_+^n ⊂ ℤ^n$ for the set of tuples $λ = (λ_1, \ldots, λ_n)$ of integers with $λ_1 ≥ \cdots ≥ λ_n$.

## 2 Automorphic forms on $GL_n(𝔸_F)$

In this section we define the class of automorphic forms whose attached Galois representations we wish to study. Let $F$ be a CM imaginary field with totally real subfield $F^+$. We say that a pair $(π, χ)$ of an automorphic representation $π$ of $GL_n(𝔸_F)$ and a continuous character $χ : (F^+)\times A_F^x → ℂ$ is RAECSDC (regular, algebraic, essentially conjugate self-dual, cuspidal) if it satisfies the following properties:

(i) $π$ is cuspidal.

(ii) $π^c ≅ π^∨ ⊗ χ \circ N_{F/F^+}$.

(iii) $χ_v(−1) = (−1)^{π}$ for each place $v|∞$ of $F^+$.

(iv) The infinitesimal character of $π_∞$ agrees with the infinitesimal character of an algebraic representation of the group $\text{Res}_{Q/ℚ} GL_n$.

We say that an automorphic representation $π$ of $GL_n(𝔸_F)$ is RACSDC if it satisfies these conditions with $χ = δ_{F/F^+}$. If $λ = (λ_τ)_{τ ∈ ℂ} ∈ (ℤ_+^n)^{\text{Hom}(F, ℂ)}$, let $Ξ_λ$ denote the irreducible representation of $GL_n^{\text{Hom}(F, ℂ)}$ which is the tensor product over $τ ∈ \text{Hom}(F, ℂ)$ of the irreducible representation of $GL_n$ with highest weight $λ_τ$. If $π_∞$ has the same infinitesimal character as $Ξ_λ$, we say that $π$ has weight $λ$.

**Theorem 2.1.** Let $(π, χ)$ be a RAECSDC automorphic representation of $GL_n(𝔸_F)$ of weight $λ$. Fix an isomorphism $ι : ℚ_l ≅ ℂ$. Then there exists a continuous semisimple representation $r_ι(π) : G_F → GL_n(ℚ_l)$, unique up to isomorphism, satisfying the following properties:
(i) \( r_i(\pi)^{c} \cong r_i(\pi)^{c} e^{1-n} \sigma(\chi) \big|_{G_F} \).

(ii) For each prime \( v \nmid l \) of \( F \), we have
\[
r_i(\pi)|_{G_{F_v}}^{F,ss} \cong r_i((\nu^{-1} \pi_v)^{c})(1-n).
\]

(iii) For each prime \( v|l \) of \( F \), \( r_i(\pi)|_{G_{F_v}} \) is de Rham. If \( \pi_v \) is unramified, then it is even crystalline. For each \( \tau : F \hookrightarrow \overline{\Q} \), we have
\[
\text{HT}_{\tau}(r_i(\pi)) = \{ \lambda_{i\tau,n}, \lambda_{i\tau,n-1} + 1, \ldots, \lambda_{i\tau,1} + n - 1 \}.
\]

Proof. The above theorem represents the culmination of the work of many people. We refer to [CH, Theorem 3.2.3] for the existence of the Galois representations, and [Car12] for the proof of local-global compatibility.

Let \( \lambda \in (\Z_n^*)^{\text{Hom(F,Q)}} \). If \( \iota \) is a choice of isomorphism \( \Q \cong C \), then we define \( \iota \lambda \in (\Z_n^*)^{\text{Hom(F,C)}} \) by the formula \( (\iota \lambda)_v = \lambda_{-1,v} \). If \( \rho : G_F \to \GL_n(\overline{\Q}) \) is a continuous representation and there exists a RAECSDC automorphic representation \( \pi \) of weight \( \iota \lambda \) such that \( \rho \cong r_i(\pi) \), we shall say that \( \rho \) is automorphic of weight \( \lambda \). This paper is dedicated to proving that certain \( l \)-adic Galois representations arise from automorphic forms in this sense. As discussed in the introduction, we must restrict to automorphic forms which are \( \iota \)-ordinary, in the sense of [Ger, Definition 5.1.2]. We do not give the definition here, but recall that it asks that for each place \( v|l \), certain Hecke operators act on a subspace of \( \nu^{-1} \pi_v \) with eigenvalues which are \( l \)-adic units. The key consequence for us is the following.

**Theorem 2.2.** Let \( (\pi, \chi) \) be a RAECSDC automorphic representation of \( \GL_n(A_F) \). Suppose that \( \pi \) is \( \iota \)-ordinary of weight \( \iota \lambda \). Then for each place \( v|l \) of \( F \), we can write
\[
r_i(\pi)|_{G_{F_v}} = \begin{pmatrix}
\psi_1 & * & * & * \\
0 & \psi_2 & * & * \\
: & \ddots & \ddots & * \\
0 & \ldots & 0 & \psi_n
\end{pmatrix}
\]
in a suitable basis. Here \( \psi_i : G_{F_v} \to \overline{\Q}_l^* \) is a continuous character satisfying the identity
\[
\psi_i(\sigma) = \prod_{\tau : F_v \hookrightarrow \overline{\Q}} \tau(\text{Art}_{F_v}(\sigma))^{-(\lambda_{\nu^{-1}i,v} + i - 1)}
\]
for all \( \sigma \) in a suitable open subgroup of \( I_{F_v} \).

Proof. This follows from local-global compatibility at \( l = p \) in the guise of [BLGGTa, Theorem A] (cf. the proof of [Ger, Corollary 2.7.8], which treats the case where \( \pi_v \) is unramified).

A representation \( \rho : G_F \to \GL_n(\overline{\Q}) \) satisfying the conclusion of this theorem will be called **ordinary** of weight \( \lambda \).

### 3 Deformation theory

In this paper we will make use of the framework for the deformation theory of conjugate self-dual Galois representations established in [CHT08], and its modification by Geraghty [Ger] to the context of ordinary Galois representations. We begin by recalling the definition of the group \( \mathcal{G}_n \) of [CHT08].
3.1 The group $G_n$

We recall that $G_n$ is the group over $\mathbb{Z}$ defined as the semi-direct product

$$G_n = (\text{GL}_n \times \text{GL}_1) \rtimes \{1, j\} = G^0_n \rtimes \{1, j\},$$

where $j$ acts on $\text{GL}_n \times \text{GL}_1$ by $j(g, \mu)j^{-1} = (\mu'g^{-1}, \mu)$. It has a representation $\text{ad}$ on $\text{Lie} \text{GL}_n = \mathfrak{gl}_n$, given by the formulae

$$\text{ad}(g, \mu)(X) = gXg^{-1}, \text{ad}(j)(X) = -tX,$$

and a character $\nu : G_n \to \text{GL}_1$ given by the formulae

$$\nu(g, \mu) = \mu, \nu(j) = -1.$$

If $\Gamma$ is a group, $R$ is a ring and $r : \Gamma \to G_n(R)$ is a homomorphism, then we write $\text{ad}r$ for the representation of $\Gamma$ on $\mathfrak{gl}_n(R)$, and $\nu \circ r$ for the induced character $\Gamma \to \text{GL}_1(R) = R^\times$.

Now suppose that $\Gamma = \Delta \times \{1, c\}$ is a group. The following result follows immediately from [CHT08, Lemma 2.1.1].

**Lemma 3.1.** Let $R$ be a ring. There is a natural bijection between the following two sets:

- The set of homomorphisms $r : \Gamma \to G_n(R)$ such that $r^{-1}(G^0_n(R)) = \Delta$.
- The set of triples $(\rho, \mu, \langle \cdot, \cdot \rangle)$, where $\rho : \Delta \to \text{GL}_n(R)$, $\mu : \Gamma \to R^\times$ are homomorphisms and $\langle \cdot, \cdot \rangle : R^\times \times R^n \to R$ is a perfect $R$-linear pairing such that for all $x, y \in R^n$ and $\delta \in \Delta$,

$$\langle x, y \rangle = -\mu(c)(y, x) \text{ and } \langle \rho(\delta)x, \rho(\delta^c)y \rangle = \mu(\delta)\langle x, y \rangle.$$

Under this correspondence we have $\mu = \nu \circ r$ and $(x, y) = txA^{-t}t$, where $r(c) = (A, -\mu(c))$.

The following definition is [CHT08, Definition 2.1.6].

**Definition 3.2.** Let $k$ be a field, and let $r : \Gamma \to G_n(k)$ be a homomorphism such that $r^{-1}(G^0_n(k)) = \Delta$. We say that $r$ is Schur if all irreducible $\Delta$-subquotients of $k^n$ are absolutely irreducible and if for all $\Delta$-invariant subspaces $k^n \supset W_1 \supset W_2$ with $k^n/W_1$ and $W_2$ irreducible, we have

$$(k^n/W_1)^c \not\cong W_2^c \otimes (\nu \circ r).$$

The following lemma follows immediately from [CHT08, Lemma 2.1.7] (and its proof).

**Lemma 3.3.** Suppose that $r : \Gamma \to G_n(k)$ is Schur. Then:

(i) $r|_{\Delta}$ is semisimple and multiplicity free, and each irreducible constituent $\rho$ satisfies $\rho^c \cong \rho^c \otimes (\nu \circ r)$.

(ii) Suppose that $r' : \Gamma \to G_n(k)$ is another Schur homomorphism such that $\text{tr} r'|_{\Delta} = \text{tr} r|_{\Delta}$, and that $k$ is algebraically closed. Then $r$ and $r'$ are $\text{GL}_n(k)$-conjugate.

(iii) Suppose that the characteristic of $k$ is not 2. Then $H^0(\Gamma, \text{ad}r) = 0$.

**Lemma 3.4.** Let $k$ be a field and $\rho = \oplus_{i=1}^s \rho_i : \Gamma \to \text{GL}_n(k)$ a semisimple representation, with each $\rho_i$ absolutely irreducible. Suppose that there is a character $\mu : \Gamma \to k^\times$ such that:

(i) For each $i = 1, \ldots, s$, there is a perfect pairing $\langle \cdot, \cdot \rangle_i$ such that $\langle x, y \rangle_i = -\mu(c)\langle y, x \rangle_i$ and $\langle \rho(\delta)x, \rho(\delta^c)y \rangle_i = \mu(\delta)\langle x, y \rangle_i$ for all $x, y \in \rho_i, \delta \in \Delta$.

(ii) For each $i \neq j$ we have $\rho_i \not\cong \rho_j$ and $\rho_i^c \not\cong \rho_j^c \otimes \mu$.

Then $\rho$ admits an extension to a homomorphism $r : \Gamma \to G_n(k)$ such that $r^{-1}(G^0_n(k)) = \Delta$. The set of $\text{GL}_n(k)$-conjugacy classes of such extensions is a principal homogeneous space for the group $\prod_{i=1}^s k^\times/(k^\times)^2$, with action given as follows. Write $r_i(c) = (A, \mu)$. Then $A = \oplus_{i=1}^s A_i$ is a block diagonal matrix, and $(\alpha_i) \in \prod_{i=1}^s k^\times$ acts by $A_i \mapsto \alpha_i A_i$. Moreover, every choice of extension is Schur.

**Proof.** The proof is an easy generalization of the proof of [CHT08, Lemma 2.1.4].
3.2 Deformation of Galois representations

Let $F$ be an imaginary CM field with totally real subfield $F^+$, and let $l$ be an odd prime. We fix a finite set of places $S$ of $F^+$ which split in $F$ and write $F(S)$ for the maximal extension of $F$ unramified outside $S$. We suppose that $S$ contains the set $S_l$ of primes of $F^+$ dividing $l$. We write $G_{F^+,S} = \text{Gal}(F(S)/F^+)$ and $G_{F,S} \subset G_{F^+,S}$ for the subset of elements fixing $F$. For each $v \in S$ we choose a place $\tilde{v}$ of $F$ above it, and write $\tilde{S}$ for the set of these places. We choose a complex conjugation $c \in G_{F^+,S}$.

We fix also a finite field $\mathbb{F}$ of characteristic $l$ and a representation $\tau: G_{F^+,S} \to G_n(k)$ such that $G_{F,S} = \tau^{-1}(\text{GL}_n \times \text{GL}_1(k))$. Let $K$ be a finite extension of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$ with ring of integers $\mathcal{O}$, maximal ideal $\lambda$, and residue field $k$. Choose a character $\chi: G_{F^+,S} \to \mathcal{O}^\times$ such that $\nu \circ \tau = \chi$.

If $v | l$ write $\Lambda_v$ for the completed group algebra $\mathcal{O}[[I_{\mathbb{F}}^v(l)^n]]$, where $I_{\mathbb{F}}^v(l)$ denotes the maximal pro-$l$ quotient of the inertia group of the maximal abelian extension of $F_{\tilde{v}}$. By class field theory, this is group isomorphic to the maximal pro-$l$ quotient of $\mathcal{O}^\times_{\mathbb{F}}$. Let $\Lambda = \widehat{\otimes}_v \Lambda_v$, the tensor product being over the primes $v | l$ of $F^+$. If $R$ is a complete Noetherian local ring with residue field $k$, then we write $C_R$ for the category of complete Noetherian local $R$-algebras with residue field $k$. We will consider deformations of $\tau$ to objects of $C_{\Lambda}$. If $v \in S$, we write $\tau|_{G_{\mathbb{F}}}$ for the composite

$$G_{\mathbb{F}} \to G_{F,S} \to G_n^0(k) \to \text{GL}_n(k).$$

**Definition 3.5.** If $v \in S - S_l$ then we define a local deformation problem at $v$ to be a subfunctor $D_v$ of the functor of all liftings of $\tau|_{G_{\mathbb{F}}}$ to objects of $C_{\mathcal{O}}$ satisfying the following conditions:

(i) $(k,\tau) \in D_v$.

(ii) Suppose that $(R_1, r_1)$ and $(R_2, r_2) \in D_v$, that $I_1$ (resp. $I_2$) is a closed ideal of $R_1$ (resp. $R_2$) and that $f : R_1/I_1 \to R_2/I_2$ is an isomorphism in $C_{\mathcal{O}}$ such that $f(r_1 \mod I_1) = r_2 \mod I_2$. Let $R_3$ denote the subring of $R_1 \times R_2$ consisting of pairs with the same image in $R_1/I_1 \cong R_2/I_2$. Then $(R_3, r_1 \times r_2) \in D_v$.

(iii) If $(R_j, r_j)$ is an inverse system of elements of $D_v$ then

$$(\lim R_j, \lim r_j) \in D_v.$$

(iv) $D_v$ is closed under equivalence.

(v) If $R \subset S$ is an inclusion in $C_{\mathcal{O}}$ and if $r: G_F \to \text{GL}_n(R)$ is a lifting of $\tau$ such that $(S, r) \in D_v$ then $(R, r) \in D_v$.

On the other hand, for $v \in S_l$ we define a local deformation problem at $v$ to be a subfunctor $D_v$ of the functor of all liftings of $\tau|_{G_{\mathbb{F}}}$ to objects of $C_{\Lambda_v}$ satisfying the same conditions with the category $C_{\mathcal{O}}$ replaced by $C_{\Lambda_v}$.

**Definition 3.6.** A lifting of $\tau$ (resp. $\tau|_{G_{\mathbb{F}}}$) to an object $R$ of $C_{\Lambda}$ is a continuous homomorphism $r : G_{F^+,S} \to G_n(R)$ (resp. $r : G_{F,S} \to \text{GL}_n(R)$) with $r \mod m_R = \tau$ (resp. $\tau|_{G_{\mathbb{F}}}$) and $\nu \circ r = \chi$ (resp. no further condition). Two liftings are said to be equivalent if they are conjugate by an element of $1 + M_n(m_R) \subset \text{GL}_n(R)$. An equivalence class of liftings is called a deformation.

Let $T \subset S$. By a $T$-framed lifting of $\tau$ to $R$ we mean a tuple $(r; \alpha_v)_{v \in T}$ where $r$ is a lifting of $\tau$ and $\alpha_v \in 1 + M_n(m_R)$. We call two framed liftings $(r; \alpha_v)$ and $(r'; \alpha'_v)$ equivalent if there is an element $\beta \in 1 + M_n(m_R)$ with $r' = \beta r \beta^{-1}$ and $\alpha'_v = \beta \alpha_v$. By a $T$-framed deformation of $\tau$ we mean an equivalence class of framed liftings.

Given a collection of deformation problems $D_v$ for $v \in S$, we have a (global) deformation problem consisting of the following data:

$$S = \left(\mathcal{F}/\mathcal{F}^+, S, \overline{S}, \Lambda, \tau, \chi; \{D_v\}_{v \in S}\right).$$
Definition 3.7. Let $T \subset S$. We call a $T$-framed lifting $(r; \alpha_v)_{v \in T}$ of $\tau$ of type $S$ if for all $v \in S$, the restriction $r|_{G_{F_v}}$ lies in $D_v$. We say that a $T$-framed deformation is of type $S$ if some (equivalently any) element of the equivalence class is of type $S$.

We let $\text{Def}^{S,T}_{\tau}$ denote the functor which associates to an object $R$ of $\mathcal{C}_\Lambda$ the set of all $T$-framed deformations of $\tau$ to $R$ of type $S$. If $T = S$ then we refer to framed deformations and write $\text{Def}^S_{\tau}$. If $T = \emptyset$ we refer to deformations and write $\text{Def}_S$.

If $R_v$ denotes the ring representing the local deformation problem $D_v$, then we write

$$R^\text{loc}_{S,T} = \widehat{\bigotimes}_{v \in T} R_v,$$

the completed tensor product being over $\mathcal{O}$. Note that $R^\text{loc}_{S,T}$ is naturally a $\Lambda$-algebra whenever $T$ contains $S_l$.

Proposition 3.8. Suppose that $\tau$ is Schur. Then the functors $\text{Def}^{S,T}_{\tau}, \text{Def}^S_{\tau}, \text{Def}_S$ are represented by objects of $\mathcal{C}_\Lambda$. We write respectively $R^\tau_{S,T}, R^S_{S}$ and $R^\text{univ}_S$ for the representing objects.

Proof. We prove this for $\text{Def}^S_{\tau}$, the other cases being similar. Consider the deformation problem in the sense of [CHT08]

$$S' = \left( F/F^+, S, \widetilde{S}, \mathcal{O}, \tau, \chi, \{ D'_{v/|}\}_{v \in S, v \notdivides |}, \bigcup \{ D'_v \}_{v \in S, v \notdivides |} \right),$$

where for $v|l$ we take $D'_v$ to be the unrestricted functor of liftings of $\tau|_{G_{F_v}}$ to objects of $\mathcal{C}_\mathcal{O}$. This is known to be represented by an object $R^\tau_{S}$ of $\mathcal{C}_\mathcal{O}$. $D'_v$ is represented by $R^\tau_v \in \mathcal{C}_\mathcal{O}$. On the other hand, for $v|| l$, $D'_v$ is represented by an object $R_e$ of $\mathcal{C}_\Lambda$, and we have a canonical homomorphism $R^\tau_v \to R_e$. The functor $\text{Def}^S_{\tau}$ is represented by $R^\tau_{S} \otimes_{\Delta_{\mathcal{O}||}} R^\tau_e \left( \bigotimes_{v||} R_v \right)$ with its induced $\Lambda$-algebra structure. □

Proposition 3.9. Suppose that $\tau$ is Schur, and let $T = S$. Then the ring $R^\tau_{S,T}$ can be presented as a quotient of a power series ring over $R^\text{loc}_{S,T}$ in $y$ variables by $r$ relations, where

$$g - r = -\dim_k H^0(G_{F^+,S}, \text{ad}\, \tau(1)) - \sum_{v||} n(n + \chi(e_v))/2.$$

Proof. This follows from [CHT08] Corollary 2.3.5) by the same argument as in the proof of Proposition 3.8. □

3.3 Local deformation problems

In this section we define some useful local deformation problems. We shall always use the notation $R^\tau_v$ for the ring representing the local deformation problem consisting of all liftings of $\tau|_{G_{F_v}}$. Thus $R^\tau_v$ is an $\mathcal{O}$-algebra (resp. a $\Lambda_e$-algebra) when $v || l$ (resp. when $v|| l$). We recall (cf. [BLGHT1]) that to give a local deformation problem, it suffices to give a quotient $R$ of the unrestricted universal lifting ring $R^\text{univ}_v$ of $\tau|_{G_{F_v}}$ which is reduced, and such that the defining ideal $I$ of $R$ is invariant under the natural conjugation action of $1 + M_n(\mathfrak{m}_{R^\tau_v})$ and satisfies $I \neq \mathfrak{m}_{R_v}$.

3.3.1 Unrestricted deformations

Proposition 3.10. Suppose that $v || l$ and that $\tau|_{G_{F_v}}$ is unramified, and that $H^0(G_{F_v}, \text{ad}\, \tau(1)) = 0$. Then $R^\tau_v$ is formally smooth over $\mathcal{O}$ of dimension $1 + n^2$.

3.3.2 Ordinary deformations

Suppose that $v \in S$. Recall that we have defined $\Lambda_v = \mathcal{O}[[F^b_v(l)]^n]$, a completed group algebra. This algebra comes equipped with the universal characters $\psi^v_i : F^b_v \to \Lambda^\times_v$, $i = 1, \ldots, n$. We assume that $K$ contains the images of $F_v$ under all embeddings $F_v \hookrightarrow \overline{\mathbb{Q}}_l$.

Suppose that $\tau|_{G_{F_v}}$ is trivial. Then Geraghty has defined a quotient $R^\Lambda_v$ of the universal lifting ring $R^\tau_v$ satisfying the following condition (cf. [Ger] Lemma 3.1.3):
• Let $E/K$ be a finite extension with ring of integers $\mathcal{O}_E$, and fix a map $\Lambda_v \to \mathcal{O}_E$ of $\mathcal{O}$-algebras. Let $\rho : G_{F_v} \to \text{GL}_n(\mathcal{O}_E)$ be a continuous lifting of $\pi|_{G_{F_v}}$. Then the map $R^\Delta_v \to \mathcal{O}_E$ classifying $\rho$ factors through $R^\Delta_v$ if and only if $\rho$ is GL$_n(\mathcal{O}_E)$-conjugate to an upper-triangular representation satisfying the following condition: if $(\chi_1, \ldots, \chi_n)$ are the characters appearing on the diagonal, then the tuple of characters $(\chi_1|_{I_{F_v}}, \ldots, \chi_n|_{I_{F_v}})$ is equal to the pushforward of the universal tuple $(\psi_1^v, \ldots, \psi_n^v)$ along the map $\Lambda_v \to \mathcal{O}_E$.

We briefly recall the construction. Let $F$ denote the $\mathcal{O}$-scheme of full flags in $\mathcal{O}^n$, and let $G_v$ denote the closed subscheme of $F \otimes_\mathcal{O} R^\Delta_v$ whose $A$-points for an $\mathcal{O}$-algebra $A$ are pairs $((\text{Fil}^i_v, \phi), v)$, where $\phi : R^\Delta_v \to A$ is an $\mathcal{O}$-algebra homomorphism and $\text{Fil}^i_v$ is an increasing filtration of $A^n$ by $A$-direct summands which are preserved by the pushforward of the universal lifting under $\phi$, and such that the action of $\rho|_{I_{F_v}}$ on $\text{gr}^i \text{Fil}^i_v = \text{Fil}^i_v / \text{Fil}^{i-1}_v$ is given by the pushforward of the universal character $\psi^v_i$ under the homomorphism $\Lambda_v \to R^\Delta_v \to A$. We thus have a projective morphism $\pi : G_v \to R^\Delta_v$, and $R^\Delta_v$ is defined as the maximal reduced, $\mathcal{O}$-flat quotient of the scheme-theoretic image of $\pi$.

Lemma 3.11. (i) The ring $R^\Delta_v[1/l]$ is formally smooth over $K$ at those closed points whose image in $\Lambda_v[1/l]$ corresponds to a tuple of characters $\psi^v_i : I_{F_v} \to E_x$ such that for all $1 \leq i < j \leq n$, $\psi^v_i \neq \psi^v_j, e\psi^v_j$. The fibers of Spec $R^\Delta_v[1/l] \to \text{Spec} \Lambda_v[1/l]$ above such points are regular and irreducible of dimension $[F_v : \mathbb{Q}][n(n-1)/2 + n^2]$. 

(ii) Suppose that $[F_v : \mathbb{Q}] > n(n-1)/2 + 1$. Then $G_v$ is $\mathcal{O}$-flat and reduced, and for each minimal prime $Q_v \subset \Lambda_v$, $G_v \otimes_{\Lambda_v} \Lambda_v/Q_v$ is $\mathcal{O}$-flat and integral, and $G_v \otimes_{\Lambda_v} \Lambda_v/(Q_v, \lambda)$ is integral. The space $\text{Spec} R^\Delta_v/Q_v$ is irreducible of dimension $[F_v : \mathbb{Q}][n(n-1)/2 + n^2 + 1]$. 

We observe that when the hypothesis of the second part of the lemma is satisfied, we can make the following further characterization of the points of $R^\Delta_v$. Let $R \in \mathcal{O}_E$ be integral. Let $E = \text{Frac}(R)$ and choose an algebraic closure $\overline{E}$ of $E$. Then a homomorphism $R^\Delta_v \to R$ factors through the quotient $R^\Delta$ if and only if the following condition is satisfied:

• Let $\rho : G_{F_v} \to \text{GL}_n(R)$ denote the induced lifting of $\pi|_{G_{F_v}}$. There exists an increasing filtration $0 = \text{Fil}^0_v \subset \text{Fil}^1_v \subset \cdots \subset \text{Fil}^n_v = \overline{E}^n$ of $\rho|_{G_{F_v}}$-invariant subspaces, such that each graded piece $\text{gr}^i \text{Fil}^i_v = \text{Fil}^i_v / \text{Fil}^{i-1}_v$ is one-dimensional, and the action of $I_{F_v}$ on this graded piece is given by the specialization of the universal character $\psi^v_i : I_{F_v} \to \Lambda_v^\times$ via the homomorphism $\Lambda_v \to R^\Delta_v \to \overline{E}$.

Indeed, the lemma shows that the scheme-theoretic image of $\pi$ is already $\mathcal{O}$-flat and reduced, so that a Spec $R$-point of the scheme-theoretic image of $\pi$ necessarily factors through Spec $R^\Delta$.

Proof. The first sentence of the first part follows from [Ger] Lemma 3.2.3 and the fact that when the characters corresponding to a closed point $x \in \text{Spec} R^\Delta_v[1/l]$ are pairwise distinct, there is a unique closed point of $G_v$ lying above it. The second sentence can be proved in the same way, using the analogue of [Ger] Lemma 3.2.3 where the diagonal characters are fixed.

We now treat the second part. Suppose that $d_v = [F_v : \mathbb{Q}] > n(n-1)/2 + 1$. We study $G_v$ by means of a finite type model. We treat the case where $F_v$ contains an $l^{th}$ root of unity, the other case being similar (and simpler). Then the pro-$l$ group $G_{F_v}(l)$ admits a presentation as a quotient of the free pro-$l$ group on generators $x_1, \ldots, x_{d_v+2}$ by the single relation

$$x_1^s[x_1, x_2][x_3, x_4] \cdots [x_{d_v+1}, x_{d_v+2}] = 1,$$

where $s$ is the largest integer such that $F_v$ contains a root of unity of order exactly $l^s$ (see [NSW00] Theorem 7.5.8). (We note that in the above situation, $d_v$ is necessarily even.) We write GL$_n$ for the $\mathcal{O}$-scheme of invertible $n \times n$ matrices, and $B_n, U_n$ for its subgroups of upper-triangular and upper-triangular unipotent matrices. Let $\mathcal{N}$ denote the $\mathcal{O}$-scheme of tuples $(g_1, \ldots, g_{d_v+2}) \in B_n^{d_v+2}$ satisfying the above relation.

Specifying a minimal prime $Q_v \subset \Lambda_v$ is the same as choosing roots of unity $\zeta_1, \ldots, \zeta_n \in \mu_v(\mathcal{O})$, the prime being given by

$$(\psi^v_1(x_1) - \zeta_1, \ldots, \psi^v_n(x_1) - \zeta_n).$$
We suppose such a choice has been fixed, and write \(\mathcal{N}_{Q_v}\) for the closed subscheme of \(\mathcal{N}\) where the diagonal entries of \(g_1\) are given by \(\zeta_1, \ldots, \zeta_n\). For \(i > 2\) let \(Z_i \subset \mathcal{N}_{Q_v} \otimes \mathcal{O} k\) denote the open subscheme where \(g_i\) has distinct eigenvalues. Let \(f_i : Z_i \to \mathbb{N} \times \mathbb{B}_{d_v}^n\) denote the projection which forgets \(g_{i+1}\) (if \(i\) is odd) and \(g_{i-1}\) (if \(i\) is even). Let \(Z\) denote the union of these open subschemes. The fibers of \(f_i\) are sets of solutions \(h \in \mathbb{B}_n\) to equations \(hgh^{-1} = gu\), where \(g \in \mathbb{B}_n\) and \(u \in \mathbb{U}_n\) and \(g\) has distinct eigenvalues. In particular, the non-empty fibers of \(f\) are smooth of dimension \(n\), being torsors for the torus \(Z\mathbb{B}_n\). On the other hand, \(f_i\) is a relative complete intersection, since \(Z_i\) is obtained by imposing \(\dim U_n\) relations on the fiber product over \(k\) of the base of \(f_i\) and \(\mathbb{B}_n\). Therefore \(f_i\) is smooth and \(Z\) is a smooth open subscheme of dimension \(\dim U_n + d_v \leq \dim \mathbb{B}_n + n = (d_v + 1) \dim \mathbb{B}_n\).

Since \(\mathcal{N}_{Q_v} \otimes \mathcal{O} k\) is globally cut out inside \(\mathbb{N} \times \mathbb{B}_{d_v}^n\) by \(\dim \mathbb{U}_n\) equations, every irreducible component has dimension at least \((d_v + 1) \dim \mathbb{B}_n\). The complement of \(Z\) is contained in the closed subscheme of \(\mathbb{N} \times \mathbb{B}_{d_v}^n\) where none of \(g_1, \ldots, g_{d_v+2}\) has distinct eigenvalues, which has dimension \(\dim \mathbb{U}_n + (d_v + 1) \dim \mathbb{B}_n - d_v\). Provided, therefore, that

\[
\dim \mathbb{U}_n + (d_v + 1) \dim \mathbb{B}_n - d_v < (d_v + 1) \dim \mathbb{B}_n - 1,
\]

or equivalently that \(d_v > n(n-1)/2+1\), we see that \(Z\) is dense in \(\mathcal{N}_{Q_v} \otimes \mathcal{O} k\), with complement of codimension at least 2. Under our hypotheses, therefore, we see that \(\mathcal{N}_{Q_v} \otimes \mathcal{O} k\) is a normal complete intersection. A similar argument shows that \(\mathcal{N}_{Q_v}\) itself is a normal complete intersection, and that \(\mathcal{N}\) is a complete intersection. Since it is also generically reduced, it is reduced.

Let \(x \in \mathcal{G}_v\) be a closed point, which therefore maps to the closed point of \(R_v^\Delta\). The completed local ring at \(\mathcal{G}_v\) is formally smooth over the completed local ring of \(\mathcal{N}\) at the point in its special fiber where all the matrices are equal to the identity (cf. the proof of [Ger, Lemma 3.2.1]), and hence is itself normal. It follows that \(\mathcal{G}_v\) is itself an \(\mathcal{O}\)-flat reduced complete intersection. A similar argument now shows that \(\mathcal{G}_v \otimes \Lambda_v, \Lambda_v/(Q_v, \lambda)\) is integral.

If \(1 \leq i < j \leq n\), we write \(I(i, j, v) \subset \Lambda_v\) for the ideal generated by the relations \(\psi_i^v(\sigma) - \psi_j^v(\sigma)\), \(\sigma \in I_{F_v}(l)\). Let \(I_v = \prod_{i,j} I(i, j, v)\). Thus \(\dim \Lambda_v/I_v = 1 + (n - 1)[F_v : Q_l]\).

**Lemma 3.12.** Let \(U \subset \text{Spec} \Lambda_v\) denote the complement of \(V(I_v)\). Then the map \(\pi_U : \mathcal{G}_{v, U} \to R_{v, U}^\Delta\) is an isomorphism.

**Proof.** It is easy to see that \(\pi_U\) is proper and quasi-finite, hence finite. It therefore suffices to check that \(\pi_U\) induces surjections on completed local rings, and this can be proved following e.g. the proof of [CHT08, Lemma 2.4.6].

We note as a particular consequence of the lemma that if \([F_v : Q_l] > n(n - 1)/2 + 1\) then for each minimal prime \(Q_v \subset \Lambda_v, R_{v, U}^\Delta/(Q_v, \lambda)\) is reduced (since \(\mathcal{G}_v \otimes \Lambda_v, \Lambda_v/(Q_v, \lambda)\) is reduced).

### 3.3.3 Level raising deformations

Now suppose that \(v \in S - S_1\). We recall a local deformation problem from [Tay08]. Choose characters

\[
\chi_{v, 1}, \ldots, \chi_{v, n} : \mathcal{O}_{F_v}^\times \to \mathcal{O}^\times,
\]

necessarily of finite order, which become trivial on reduction modulo \(\lambda\). Suppose that \(q_v \equiv 1 \mod l\), and that \(\pi|_{\mathcal{G}_F}\) is trivial. We write \(\mathcal{D}_v^\times\) for the functor of liftings \(\rho\) of \(\pi|_{\mathcal{G}_F}\) to objects of \(\mathcal{C}_\mathcal{O}\) such that for all \(\sigma \in I_{F_v}\), we have

\[
\text{char}_\rho(\sigma)(X) = \prod_{i=1}^n (X - \chi_{v, i}(\text{Art}_{F_v}(\sigma))^{-1}).
\]

This defines a local deformation problem. Write \(R_{v, v}^\times\) for the corresponding local lifting ring. We have the following (cf. [Tho12, Proposition 3.16]):
Proposition 3.13. (i) Suppose that $\chi_{v,j} = 1$ for each $j$. Then each irreducible component of $R_v^1$ has dimension $n^2 + 1$, and every prime of $R_v^1$ minimal over $\lambda$ contains a unique minimal prime. Every generic point is of characteristic zero.

(ii) Suppose that the $\chi_{v,j}$ are pairwise distinct. Then Spec $R_v^{n^2}$ is irreducible of dimension $n^2 + 1$, and its generic point is of characteristic zero. Moreover, Spec $R_v^{n^2}[1/l]$ is formally smooth over $K$.

Proposition 3.14. Suppose that the $\chi_{v,j}$ are pairwise distinct. Let $A$ be an $O$-flat complete Noetherian local $O$-algebra. (We do not assume that the residue field of $A$ is $k$.) Suppose that $\text{Spec } A[1/l]$ is connected. Then Spec$(A \otimes_{O} R_v^{n^2})[1/l]$ is connected.

Proof. In the case that $A = O$, this is [Tay08] Lemma 3.4. In fact, the argument given there shows, in our case, that any maximal ideal of $(A \otimes_{O} R_v^{n^2})[1/l]$ lies in the same connected component of Spec$(A \otimes_{O} R_v^{n^2})[1/l]$ as a maximal ideal of $\text{Spec } A[1/l]$, where we make $A[1/l]$ a quotient of $(A \otimes_{O} R_v^{n^2})[1/l]$ using the map $R_v^{n^2} \to O$ classifying the lifting of $\pi|_{G_{F_v}}$ sending a choice of Frobenius lift to the identity matrix and with restriction to inertia given by $\text{diag}(\chi_{v,1}, \ldots, \chi_{v,n})$. The result now follows on applying the fact that Spec $A[1/l]$ is connected and that $(A \otimes_{O} R_v^{n^2})[1/l]$ is a Jacobson ring. □

3.3.4 Steinberg deformations

Now suppose that $v \in S - S_1$ and that $\pi|_{G_{F_v}}$ is trivial and that $q_v \equiv 1 \mod l$. Then in [Tay08] §3 is defined a quotient $R_v^{\text{St}} = R_v^2/I_v^{\text{St},1}$ of $R_v^2$. (In fact the assumption $l > n$ is made in this reference, but this is not needed for the definition or the results that follow.) We recall that by definition, $R_v^2/I_v^{\text{St},1}$ is the quotient of $R_v^2$ defined by the condition that the characteristic polynomial of a Frobenius lift has the form $\prod_{i=1}^{n}(X - q_v^{n-i})$ for some $\alpha$. Then $R_v^2/I_v^{\text{St},1}$ is the maximal $O$-flat quotient of this ring, and defines a local deformation problem. We recall the following result from [Tay08].

Proposition 3.15. Spec $R_v^{\text{St}}$ is irreducible of dimension $n^2 + 1$, and its generic point is of characteristic zero. Moreover, Spec $R_v^{\text{St}}[1/l]$ is formally smooth over $K$.

Proposition 3.16. Let $A$ be an $O$-flat complete Noetherian local $O$-algebra. (We do not assume that the residue field of $A$ is $k$.) Suppose that $\text{Spec } A[1/l]$ is connected. Then Spec$(A \otimes_{O} R_v^{\text{St}})[1/l]$ is connected.

Proof. The proof is the essentially the same as that of Proposition 3.14 above. □

3.3.5 Taylor-Wiles deformations

Finally suppose that $q_v \equiv 1 \mod l$ and that $\pi|_{G_{F_v}}$ is unramified. Choose an eigenvalue $\pi_v$ of $\pi(\text{Frob}_v)$ of multiplicity $n_v, 1 \leq n_v \leq n$, such that $\pi(\text{Frob}_v)$ acts semisimply on its generalized $\pi_v$-eigenspace. Then we can write $\pi|_{G_{F_v}} = \pi_v \oplus \psi_v$, where $\psi_v(\text{Frob}_v)$ is equal to $\pi_v \cdot \text{Id}_{n_v}$. We define $D_v^{\text{TW}}(\pi_v)$ to be the functor of lifts $\rho = s \oplus \psi$, where this decomposition lifts the previous one and $s$ is unramified and $\psi$ may be ramified, but the restriction to inertia is scalar. Then $D_v^{\text{TW}}(\pi_v)$ is a local deformation problem (cf. [Tho12] Lemma 4.2]).

Fix a deformation problem

$$S = \left( F/F^+, S, \tilde{S}, \Lambda, \tau, \chi, \{D_v\}_{v \in S} \right),$$

a positive integer $N$, and a finite set $Q_N$ of primes $v$ of $F^+$ split in $F$, disjoint from $S$. Choose for each $v \in Q_N$ a prime $\tilde{v}$ of $F$ above it, and let $Q_N = \{ \tilde{v} \mid v \in Q_N \}$. We suppose that $Q_N$ has $q$ elements and that for each $v \in Q_N$, we have $q_v \equiv 1 \mod l^N$. Choose for each $v \in Q_N$ an eigenvalue $\pi_v$ of $\pi(\text{Frob}_v)$ as above, so that $D_v^{\text{TW}}(\pi_v)$ is defined. We refer to the tuple $(Q_N, \tilde{Q}_N, \{\pi_v\}_{v \in Q_N})$ as a choice of Taylor-Wiles data of order $q$ and level $N$.

In this case we define an auxiliary deformation problem

$$S_N = \left( F/F^+ \cup Q_N, \tilde{S} \cup \tilde{Q}_N, \Lambda, \tau, \chi, \{D_v\}_{v \in \tilde{S} \cup \tilde{Q}_N} \cup \{D_v^{\text{TW}}(\pi_v)\}_{v \in Q_N} \right).$$
This is an abuse of notation since $S_N$ depends on the choice of Taylor-Wiles data rather than just $N$. However in practice we will use only a single choice of Taylor-Wiles data for each integer $N$, so we hope that this will not cause confusion.

**Lemma 3.17.** With the above choices, let $\Delta_N$ denote the maximal $l$-power order quotient of $\prod_{v \in Q_S} k(v)^\times$. Suppose that $\tau$ is Schur, so that $R^\text{univ}_S$ is defined. Then $R^\text{univ}_S$ has a canonical $\mathcal{O}[\Delta_N]$-algebra structure, and the natural surjection $R^\text{univ}_S \to R^\text{univ}_S$ induces an isomorphism

$$R^\text{univ}_S/\mathfrak{a}_N \cong R^\text{univ}_S,$$

where $\mathfrak{a}_N \subset \mathcal{O}[\Delta_N]$ is the augmentation ideal.

### 3.4 Pseudodeformations of Galois representations

In this section we consider pseudodeformations of $\tau$. Since we do not wish to exclude the case $l < n$, we need to use group determinants rather than pseudocharacters. We follow here the exposition of [Che]. We begin by recalling the relevant definitions.

**Definition 3.18.** Let $A$ be a ring, and $M, N$ be $A$-modules. Then $M$ (resp. $N$) defines a functor $M$ (resp. $N$) : $A$-alg $\to$ Sets by the rule $B \mapsto M \otimes_A B$ (resp. $B \mapsto N \otimes_A B$). An $A$-polynomial law from $M$ to $N$ is a natural transformation $P : M \to N$. Such an $A$-polynomial law is said to be homogeneous of degree $n$ if for all $B \in A$-alg, $b \in B$, and $m \in M \otimes_A B$, we have $P(bm) = b^n P(m)$.

If $G$ is a group then an $A$-valued determinant on $G$ of dimension $n$ is an $A$-polynomial law $D : \mathbb{A}[G] \to A$, homogeneous of degree $n$, which is multiplicative, in the sense that $D(1) = 1$ and for all $B \in A$-alg, $r, s \in B[G] = \mathbb{A}[G] \otimes_A B$, we have $D(rs) = D(r) \cdot D(s)$.

**Definition 3.19.** Given an $A$-valued determinant $D : \mathbb{A}[G] \to A$ of dimension $n$, we define the characteristic polynomial of $g \in G$ by the formula

$$D_{\mathbb{A}[G]}(t - g) = \sum_{i=0}^n (-1)^i \Lambda_i(g) t^{n-i}.$$ 

If $G$ and $A$ are endowed with topologies, we say that $D$ is continuous if the maps $\Lambda_i : G \to A$ are continuous for each $i = 0, \ldots, n$.

The following lemma collects some basic facts in the theory of group determinants.

**Lemma 3.20.** Let $G$ be a group.

(i) Let $A$ a ring, and $\rho : G \to \text{GL}_n(A)$ a representation. Extend $\rho$ to an algebra homomorphism $A[G] \to M_{n \times n}(A)$. Then the formula $D = \det \circ \rho$ defines an $A$-valued determinant on $G$ of dimension $n$.

(ii) Let $k$ be an algebraically closed field, and let $D : G \to k$ be a determinant of dimension $n$. Then there exists a semisimple representation $\rho : G \to \text{GL}_n(k)$, unique up to isomorphism, with $D = \det \circ \rho$.

**Proof.** The first part is elementary. The second part is [Che] Theorem 2.12. $\square$

We set $D = \det \circ \tau|_{G_{F,S}}$.

**Definition 3.21.** A pseudodeformation of $D$ to an object $R$ of $\mathcal{C}_\mathcal{O}$ is a continuous determinant $D : G_{F,S} \to R$ of dimension $n$ such that $D \otimes_R k = D$. (Here $R$ is given its natural profinite topology.) We write $\text{PDef}_S$ for the set-valued functor which associates to an object $R$ of $\mathcal{C}_\mathcal{O}$ the set of all pseudodeformations of $D$ to $R$.

**Proposition 3.22.** (i) The functor $\text{PDef}_S$ is represented by an object $Q_S$ of $\mathcal{C}_\mathcal{O}$.

(ii) Write $\Lambda^\text{univ}_i : G \to Q_S$ for the coefficients of the universal characteristic polynomial, and let $\mathcal{L}$ be a set of finite places of $F$, disjoint from $S$, of Dirichlet density 1. Then $Q_S$ is topologically generated as an $\mathcal{O}$-algebra by the elements $\Lambda^\text{univ}_i(\text{Prob}_w)$ for $w \in \mathcal{L}, i = 0, \ldots, n$. 


Proof. The first part follows easily from [Che] Proposition 3.3 and [Che] Proposition 3.7. For the second part, we must show that for every \( r \geq 0 \), \( Q_S / \mathfrak{m} Q_S \) is generated as an \( \mathcal{O} \)-algebra by the elements \( \Delta^\text{univ}_i \text{(Frob}_i) \). This follows from the Chebotarev density theorem and [Che] Corollary 1.14. \( \square \)

Definition 3.23. Fix a global deformation problem \( S \), and suppose that \( \pi \) is Schur. Then there is a natural map \( Q_S \otimes _{\mathcal{O}^2 \Lambda} \to R^\text{univ}_S \) classifying the determinant of the universal deformation. We write \( P_S \) for the image of this map.

By Proposition 3.22 we could have defined \( P_S \) as the \( \Lambda \)-subalgebra of \( R^\text{univ}_S \) topologically generated by the coefficients of the characteristic polynomials of elements of \( G_{F,S} \). We have followed a slightly circuitous route to its definition in order to have access to the following lemma.

Lemma 3.24. Suppose that \( \pi \) is Schur. Fix an integer \( q \geq 0 \). Then there exists an integer \( C > 0 \) depending only on \( q, \pi \) and \( S \) such that for any set \( S' \supset S \) of finite primes of \( F^+ \) split in \( S \) such that \( |S' - S| \leq q \) and any deformation problem \( S' \) unramified outside \( S' \), \( P_S \) can be written as a quotient of a power ring over \( \mathcal{O} \) in \( C \) variables.

Proof. It suffices to prove the result for \( Q_{S'} \), and hence it is enough to show that \( \dim_k P_{\text{Def}}(k[\epsilon]) \) can be bounded independently of \( S' \). Let \( F_0 \subset F' \) denote the extension of \( F^+ \) cut out by \( \pi \), and let \( F_1 \subset F' \) denote the maximal extension of \( F_0 \) which is pro-\( l \) and unramified outside \( S' \). [Che] Lemma 3.8 shows that any pseudodeformation to \( k[\epsilon] \) factors through \( \text{Gal}(F_1/F) \), and [Che] Proposition 2.38 then implies that to prove the lemma it suffices to bound the number of topological generators of \( \text{Gal}(F_1/F) \) solely in terms of \( |S' - S| \). This is an exercise in class field theory. \( \square \)

We now attempt to clarify the relation between \( P_S \) and \( R^\text{univ}_S \). Suppose that the residual representation \( \pi \) is Schur and that \( \pi|_{G_{F,S}} = \bar{\pi}_1 \oplus \bar{\pi}_2 \) is a direct sum of two absolutely irreducible representations. Then the centralizer \( Z_{\text{GL}_n(k)}(\bar{\pi}) \) is equal to \( \mu_2 \times \mu_2 \subset k^\times \times k^\times \).

Suppose that the local deformation problems defining \( S \) are amongst those defined above. The group \( \mu_2 \times \mu_2 \) then acts on the ring \( R^\text{univ}_S \) by conjugation of the universal deformation, as follows. If \( [\bar{\pi}] \) is a deformation represented by a choice of lifting \( r : G_{F,S} \to \tilde{G}_n(R) \), we can lift the decomposition \( \bar{\pi}|_{G_{F,S}} = \bar{\pi}_1 \oplus \bar{\pi}_2 \) to a decomposition \( R^\text{univ} = R^\text{univ}_1 \oplus R^\text{univ}_2 \), not necessarily \( G_{F,S} \)-invariant. This defines a lifting \( \mu_2 \times \mu_2 \to \text{GL}_n(R) \), and given \( \gamma \in \mu_2 \times \mu_2 \) we define \( \gamma r \) by \( \gamma r(\sigma) = r(\sigma)\gamma^{-1} \). This is well-defined since for any two liftings \( \gamma, \gamma' \), we have \( \gamma' \gamma^{-1} \in 1 + M_n(\mathfrak{m} R) \), so the two deformations \( [\gamma r] \) and \( [\gamma' r] \) are equal.

Proposition 3.25. (i) Suppose that \( \pi|_{G_{F,S}} \) is absolutely irreducible. Then the inclusion \( P_S \subset R^\text{univ}_S \) is an equality.

(ii) Suppose that \( \pi \) is Schur. Then the inclusion \( P_S \subset R^\text{univ}_S \) is finite.

(iii) Suppose that \( \pi \) is Schur, that \( \pi|_{G_{F,S}} = \bar{\pi}_1 \oplus \bar{\pi}_2 \), and that the local deformation problems defining \( S \) are amongst those defined above, as in the above discussion. Let \( p \subset R^\text{univ}_S \) be a dimension one prime, let \( E \) be the fraction field of \( R^\text{univ}_S / p \), and suppose that the induced representation \( r : G_{F,S} \to \tilde{G}_n(E) \) is such that \( r|_{G_{F,S}} \) is absolutely irreducible. Let \( q = P_S \cap p \). Then the group \( \mu_2 \times \mu_2 \) permutes the primes of \( R^\text{univ}_S \) above \( q \) transitively.

Proof. For the first part, it suffices to show that the map \( P_S \to R^\text{univ}_S \) induces a surjection on Zariski tangent spaces, or equivalently that if \( r_1, r_2 \) are liftings of \( \pi \) to \( k[\epsilon] \) such that for every \( x \in k[\epsilon][G_{F,S}] \), \( r_1(x) \) and \( r_2(x) \) have the same characteristic polynomial, then \( r_1 \) and \( r_2 \) are \( 1 + \epsilon M_n(k) \)-conjugate. By the corresponding result for deformations valued in \( \text{GL}_n \), we may assume that \( r_1|_{G_{F,S}} = r_2|_{G_{F,S}} \).

By Lemma 3.1 the data of \( r_1 \) is equivalent to the data of \( r_1|_{G_{F,S}} \) and a matrix \( A_1 \) realizing the conjugate self-duality of \( r_1 \), that is, satisfying the relation

\[
\chi(\delta) A_1^{-1} = r_1(\delta) A_1^{-1} r_1(\delta^\epsilon)
\]

for all \( \delta \in G_{F,S} \). If \( B \in \text{GL}_n(k[\epsilon]) \), then conjugating by \( B \) sends the matrix \( A_1 \) to \( BA_1 B^\epsilon \). By Schur’s lemma (in the guise of [CHT08] Lemma 2.1.8), \( A_1 \) and \( A_2 \) differ by a scalar. Since the characteristic is not 2, we can choose a scalar matrix in \( 1 + \epsilon M_n(k) \) which takes \( A_1 \) to \( A_2 \).
For the second part, it suffices to show that $R_S^{\text{univ}}/m_{P_{S''}} R_S^{\text{univ}}$ is Artinian. Suppose not; then there exists a dimension one prime $p \subset R_S^{\text{univ}}/m_{P_{S''}} R_S^{\text{univ}}$. Let $\bar{A}$ denote the residue ring, and $A$ the normalization of $\bar{A}$ in its fraction field $E$. Fix an isomorphism $A \cong k'[\|T\|]$, where $k'/k$ is a finite field extension.

Let $r : G_{F+S} \to G_n(A)$ denote a representative of the corresponding deformation. By construction, we have $\det \circ r|_{G_{F+S}} = \det \circ \bar{r}|_{G_{F+S}}$, hence $r|_{G_{F+S}} \cong \bar{r}|_{G_{F+S}}$ over $E$. Let $E^n = \oplus_i V_i$ denote the isotypic decomposition under $r|_{G_{F+S}} \otimes A E$, and let $L_i = A^n \cap V_i$. The map $L_i/TL_i \to A^n/TA^n$ is injective, so $\oplus_i L_i \to A^n$ is an isomorphism ($\bar{r}|_{G_{F+S}}$ is multiplicity-free).

Let $E^n = \oplus_i V_i'$ denote the isotypic decomposition with respect to $\bar{r}|_{G_{F+S}}$, and set $L_i' = V_i' \cap A^n$. Choose $\gamma \in GL_n(E)$ with $\gamma \bar{r}|_{G_{F+S}} \gamma^{-1} = \bar{r}|_{G_{F+S}}$. Then $\gamma = (\gamma_i)_i \oplus_2 \text{Hom}_{G_{F+S}}(V_i, V_i')$, and after scaling each $\gamma_i$, we can assume that $\gamma(L_i) = L_i'$, and hence $\gamma \in \text{GL}_n(A)$. Since $\bar{r} \equiv r \mod T$, we can even assume that $\gamma \in 1 + M_{n \times n}(A)$. Indeed, the reduction of $\gamma_i$ mod $m_A$ centralizes $T$, so is scalar under the identification $L_i/TL_i = L_i'/TL_i'$. After multiplying by an element of $A^\times$ we can therefore assume that $\gamma_i \equiv 1 \mod m_A$. Arguing as above, we find that we can choose $\gamma \in 1 + M_{n \times n}(A)$ and $\gamma \bar{r}^{-1} = \bar{r}$. This contradicts the universal property of $R_S^{\text{univ}}$.

For the third part, we first prove the following statement (we treat the case of $p$ of characteristic $l$ here, the mixed characteristic case being similar). Let $A = k'[\|T\|]$, where $k'/k$ is a finite extension, and let $E$ denote its fraction field. Let $r_1, r_2 : G_{F+S} \to G_n(A)$ be lifts of $\bar{r}$ such that $r_1|_{G_{F+S}} \otimes_A E$ and $r_2|_{G_{F+S}} \otimes_A E$ are absolutely irreducible and have the same characteristic polynomials. Then there exists $\gamma \in \text{GL}_n(A)$ with $\gamma r_1|_{G_{F+S}} \gamma^{-1} = r_2|_{G_{F+S}}$.

Write $A_1, A_2$ for the matrices realizing the conjugate self-duality of these representations. Since the characteristic polynomials are equal, there exists $\gamma \in \text{GL}_n(E)$ with $\gamma r_1|_{G_{F+S}} \gamma^{-1} = r_2|_{G_{F+S}}$. By the Cartan decomposition, we can write $\gamma = k_2^{-1} \gamma' k_1$ with $k_i \in \text{GL}_n(A)$ and $\gamma' = \text{diag}(T^{a_1}, \ldots, T^{a_n})$ and $a_1 \geq a_2 \geq \cdots \geq a_n$. Replacing $r_i$ with $k_i r_i k_i^{-1}$, we can suppose that $\gamma = \gamma'$. Then Schur’s lemma shows that $\gamma A_1 \gamma' = \lambda A_2$ for scalar $\lambda$.

The proof will be complete if we can show that $a_1 = \cdots = a_n$. Suppose instead that $a_1 = \cdots = a_k > a_{k+1}$. After adjoining a suitable root of $T$ to $A$ we can suppose that $\lambda$ is a unit and $a_1 + \cdots + a_n = 0$. For $\sigma \in G_{F+S}$, we write

$$r_1(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}, \quad A_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where the diagonal block matrices have size $k$ and $n-k$, respectively. If $i > k \geq j$ then $a_i - a_j < 0$, so $\gamma r_1^{-1}$ implies that $c(\sigma)$ and $D$ are divisible by $T$. Thus we have

$$\bar{r}_1(\sigma) = \begin{pmatrix} \bar{a}(\sigma) & b(\sigma) \\ 0 & \bar{d}(\sigma) \end{pmatrix}, \quad \bar{A}_1 = \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & 0 \end{pmatrix}.$$ 

Since $\bar{r}$ is Schur (and hence $\bar{r}|_{G_{F+S}}$ is semisimple) we can find a matrix $u \in M_{k \times (n-k)}(k)$ with

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{a}(\sigma) & b(\sigma) \\ 0 & \bar{d}(\sigma) \end{pmatrix} \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \bar{a}(\sigma) & 0 \\ 0 & \bar{d}(\sigma) \end{pmatrix}.$$ 

Again using the fact that $\bar{r}$ is Schur, we see that the matrix

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u^t & 1 \end{pmatrix} = \begin{pmatrix} \bar{A} + uu^t + \bar{B}u^t & \bar{B} \\ \bar{C} & 0 \end{pmatrix}$$

must be block diagonal. Since it is also non-singular, this contradiction concludes the proof of the statement.

Now suppose that $p_1, p_2$ are primes of $R_S^{\text{univ}}$ above $q$, as in the statement of the proposition. We can find a finite extension $F$ of the fraction field of $R_S/q$ with ring of integers $A$ as above and representations $r_1, r_2 : G_{F+S} \to G_n(A^0)$, where $A^0 \subset A$ is the subring of elements with image in the residue field contained inside $k$, such that the induced homomorphisms $R_S \to A^0$ have kernels $p_1, p_2$. By what we have just proved, there exists $\gamma \in \text{GL}_n(A)$ with $\gamma r_1|_{G_{F+S}} \gamma^{-1} = r_2|_{G_{F+S}}$, and hence $\gamma A_1 \gamma' = \mu A_2$ for some $\mu \in A^\times$. After possibly modifying $\gamma$ by a scalar, we see that $\gamma$ realizes the action of an element of the group $\mu_2 \times \mu_2$. This completes the proof.
3.5 Reducible deformations

In this section we suppose that \( \tau \) is Schur and that its restriction to \( G_{F,S} \) has the form \( \tau|_{G_{F,S}} = \tau_1 \oplus \tau_2 \), where the \( \tau_i \) are absolutely irreducible. Let \( n_i = \dim \tau_i \). Let \( \Lambda_i = \bigotimes_{v \in \mathcal{O}} \mathcal{O}[I_{F_v}^n(\mathcal{I}^n)] \), the completed tensor product being over \( \mathcal{O} \). Then the representations \( \tau_i \) admit unique extensions to homomorphisms \( \tau_i : G_{F^+,S} \to G_n(k) \) with the property that \( \nu \circ r_i = \nu \circ r \) and such that, writing \( \tau(c) = (A, -\chi(c))j, \tau_i(c) = (A_i, \chi(c))j \), the matrix \( A \) is block diagonal with the diagonal pieces given by \( A_1, A_2 \).

Suppose that \( r_i : G_{F^+,S} \to G_n(R), i = 1, 2 \), are liftings of \( \tau_i \) to \( R \), where \( R \in \mathcal{C}_\mathcal{O} \) is endowed with structures of \( \Lambda_1 \)- and \( \Lambda_2 \)-algebra. Suppose further that for each \( i \) we have \( \nu \circ r_i = \chi \). Then we can define the direct sum
\[
 r_1 \oplus r_2 : G_{F^+,S} \to G_n(R)
\]
in an obvious manner, and \( \nu \circ (r_1 \oplus r_2) = \chi \).

**Lemma 3.26.**

(i) Let \( A \in \mathcal{C}_A \) be a discrete valuation ring with fraction field \( E \) and residue field \( k \), and let \( r : G_{F^+,S} \to G_n(A) \) be a lifting of \( \tau \) such that \( r|_{G_{F,S}} \otimes_A E \) is reducible. Then, after possibly conjugating by an element \( r_i \in 1 + \mathfrak{m}_A(\mathfrak{m}_A) \), there is a direct sum decomposition \( r = r_1 \oplus r_2 \) lifting \( \tau = \tau_1 \oplus \tau_2 \). In particular, \( r|_{G_{F,S}} \otimes_A E \) is semisimple and multiplicity-free.

(ii) Let \( A \) be as above and let \( r : G_{F^+,S} \to G_n(A) \) be a lifting of \( \tau \) such that \( r|_{G_{F,S}} \otimes_A E \) is irreducible. Then \( r|_{G_{F,S}} \otimes_A E \) is absolutely irreducible.

(iii) Let \( R \) be an Artinian \( \mathcal{O} \)-algebra with residue field \( k \), and let \( r : G_{F^+,S} \to G_n(R) \) (resp. \( r' \)) be a lifting of \( \tau \) of the form \( r_1 \oplus r_2 \) (resp. \( r'_1 \oplus r'_2 \)), where \( r_1 \) (resp. \( r'_1 \)) lifts \( \tau_1 \). Suppose there exists \( \gamma \in \text{GL}_n(R) \) with \( \gamma \equiv 1 \mod \mathfrak{m}_R \), \( \gamma r \gamma^{-1} = r' \). Then \( \gamma \) is a block diagonal matrix, with blocks of size \( n_1, n_2 \).

**Proof.** For the first part, taking the reduction of a Jordan-Hölder filtration of \( r|_{G_{F,S}} \otimes_A E \) shows that this representation has at most two Jordan-Hölder factors and is Schur, hence semisimple. Let \( E^n = V_1 \oplus V_2 \) be a decomposition into simple submodules, and let \( L_i = V_i \cap A^n \). Let \( T \) be a uniformizer of \( \mathcal{O} \). Then the natural map \( L_i/TL_i \to A^n/TA^n \) is injective and \( G_{F,S} \)-equivariant, hence \( A^n = L_1 \oplus L_2 \). After renumbering, we can assume that the image of \( L_i/TL_i \) in \( A^n/T^n \) is \( \tau_i|_{G_{F,S}} \). Let \( e_1, \ldots, e_{n_1} \) be a basis of \( L_1 \) lifting the standard one, and let \( f_1, \ldots, f_{n_2} \) be a basis of \( L_2 \) lifting the standard one. Then in the basis \( e_1, \ldots, e_{n_1}, f_1, \ldots, f_{n_2} \), the representation \( r \) takes the desired form.

For the second part, consider the extension of \( r|_{G_{F,S}} \) to an algebra homomorphism \( A|_{G_{F,S}} \to M_{n \times n}(A) \), and let \( A \) denote the image. Write \( E = A \otimes_A E \). Then \( E \) is a simple algebra by Schur’s lemma. On the other hand, by [Che, Theorem 2.22] we can construct a family of orthogonal idempotents \( e_1 + \cdots + e_{n_1} = 1 \) in \( E \). We must therefore have \( E = M_{n \times n}(E) \), and the representation is absolutely irreducible.

Finally, for the third part we may suppose by induction on the length of \( R \) that \( r \equiv r' \) mod \( I \), where \( I \subseteq R \) is an ideal with \( \mathfrak{m}_R \cdot I = 0 \), and that \( \gamma \equiv 1 \mod I \). Then we can write \( r' = r(1 + \phi) \), \( \gamma = 1 + m \), where \( \phi \in Z^1(G_{F^+,S}, \text{ad} \tau) \otimes_k I \) and \( m = 1 + M_{n \times n}(I) \). A calculation shows that, writing \( m \) in block form as
\[
 m = \begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]
we have \( B \in \text{Hom}_{G_{F,S}}(\tau_2, \tau_1) \otimes_k I = 0, C \in \text{Hom}_{G_{F,S}}(\tau_1, \tau_2) \otimes_k I = 0 \). The result follows. \( \square \)

Fix a global deformation problem
\[
 S = \left( F/F^+, S, \tilde{S}, \Lambda, \tau, \chi, \{ D_v \}_{v \in S} \right).
\]

**Definition 3.27.** Let \( R \) be an object of \( \mathcal{C}_A \). By a reducible deformation of \( \tau \) to \( R \), we mean a deformation whose equivalence class contains a lifting of the form \( r = r_1 \oplus r_2 \), where for \( i = 1, 2 \) \( r_i : G_{F^+,S} \to G_n_i(R) \) is a lifting of \( \tau_i \). We write \( \text{Def}_S^{\text{red}} \) for the subfunctor of \( \text{Def}_S \) of reducible deformations.

**Proposition 3.28.** \( \text{Def}_S^{\text{red}} \) is a closed subfunctor of \( \text{Def}_S \), hence is represented by a quotient \( R_S^{\text{red}} \) of \( R_S^{\text{inv}} \).
Proof. We must show that $\text{Def}_S^{\text{red}} \subset \text{Def}_S$ is relatively representable. This means that for every diagram of Artinian $\mathcal{C}_A$-algebras

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\alpha \downarrow & & \downarrow \beta \\
C & \rightarrow & \\
\end{array}
\]

the diagram of sets

\[
\begin{array}{ccc}
\text{Def}_S^{\text{red}}(A \times_C B) & \rightarrow & \text{Def}_S^{\text{red}}(A) \times_{\text{Def}_S^{\text{red}}(C)} \text{Def}_S^{\text{red}}(B) \\
\downarrow & & \downarrow \\
\text{Def}_S(A \times_C B) & \rightarrow & \text{Def}_S(A) \times_{\text{Def}_S(C)} \text{Def}_S(B) \\
\end{array}
\]

is Cartesian. We are immediately reduced to showing that the top horizontal arrow is surjective. This is an easy exercise using the third part of the lemma above.

Lemma 3.29. Let $p$ be a dimension one prime of $R_S^{\text{univ}}$ not containing the kernel of the map $R_S^{\text{univ}} \rightarrow R_S^{\text{red}}$. Let $\bar{A} = R_S^{\text{univ}}/p$, and let $E = \text{Frac} \bar{A}$. Let $r : G_{F,S} \rightarrow G_n(\bar{A})$ denote the induced representation. Then $r|_{G_{F,S}} \otimes \bar{A} E$ is absolutely irreducible.

Proof. Write $A$ for the normalization of $\bar{A}$ in $E$, and let $A^0 \subset A$ denote the subring of elements whose image modulo the maximal ideal of $A$ lies in $k$. Suppose for contradiction that $r \otimes \bar{A} E$ is reducible. By the first part of Lemma 3.26 above, there is $\gamma \in 1 + M_n(\mathfrak{m}_q^p)$ such that $\gamma r \gamma^{-1}$ admits a direct sum decomposition $r = r_1 \oplus r_2$, where $r_1$ lifts $\tau_i$, and this contradicts the hypothesis on $p$. The lemma now follows from the second part of Lemma 3.26.

3.6 Twisting

We suppose in this section that $l$ is coprime to $n$. We fix the global deformation problem

\[
S = \left( F/F^+, S, S, \Lambda, \tau, \epsilon^{1-n} \delta_{L/L^+}, \{ R_v^\Lambda \}_{v \in S}, \{ R_v^{\text{St}} \}_{v \in S(B)} \cup \{ R_v^1 \}_{v \in S} \cup \{ R_v^1 \}_{v \in R} \right).
\]

Lemma 3.30. Suppose that $R \in \mathcal{C}_A$, and $r : G_{F,S} \rightarrow G_n(R)$ is a lifting of type $S$. Suppose that $\psi : G_{F,S} \rightarrow R^\times$ is a character with trivial reduction modulo $\mathfrak{m}_R$ such that $\psi \psi^c = 1$, and $\psi$ is unramified away from the places of $F$ dividing $l$. Then there exists a unique lifting $r \otimes \psi : G_{F,S} \rightarrow G_n(R)$ such that $\langle r \otimes \psi \rangle|_{G_{F,S}} = r|_{G_{F,S}} \otimes \psi$ and $r(\epsilon) = (r \otimes \psi)(\epsilon)$.

Proof. This is an immediate consequence of Lemma 3.1 and the invariance of our chosen local deformation problems under twisting by characters.

Let $\Delta$ denote the Galois group of the maximal abelian pro-$l$ extension of $F$ unramified outside $l$. We recall that $\mathcal{O}[\Delta/(c+1)]$ represents the functor of lifts of the trivial character to characters $\psi : G_{F,S} \rightarrow R^\times$ which are unramified away from the primes dividing $l$ and satisfy the relation $\psi \psi^c = 1$.

Lemma 3.31. There is a commutative diagram

\[
\begin{array}{ccc}
P_S & \rightarrow & R_S^{\text{univ}} \\
\downarrow & & \downarrow \\
P_S \hat{\otimes} \mathcal{O}[\Delta/(c+1)] & \rightarrow & R_S^{\text{univ}} \hat{\otimes} \mathcal{O}[\Delta/(c+1)]. \\
\end{array}
\]

Proof. Let $\Psi : G_{F,S} \rightarrow \mathcal{O}[\Delta/(c+1)]^\times$ denote the universal character. Let $r_S$ denote a representative of the universal deformation. Then the twist $r_S \otimes \Psi$ induces the homomorphism $P_S^{\text{univ}} \rightarrow R_S^{\text{univ}} \hat{\otimes} \mathcal{O}[\Delta/(c+1)]$ of the lemma. The map $P_S \rightarrow R_S^{\text{univ}} \hat{\otimes} \mathcal{O}[\Delta/(c+1)]$ has image in the subring $P_S \hat{\otimes} \mathcal{O}[\Delta/(c+1)]$, and this implies the result.
Let \( \psi_0 : G_{F,S} \to \mathcal{O}^\times \) denote the Teichmüller lift of \( \det \tau|_{G_{F,S}} \). We define a quotient \( R_{S,\psi_0}^{\text{univ}} \) of \( R_S^{\text{univ}} \) by the condition that \( \det r_S|_{G_{F,S}} = \psi_0 \), where \( r_S \) is a representative of the universal deformation.

**Lemma 3.32.** There is a canonical isomorphism \( R_S^{\text{univ}} \cong R_{S,\psi_0}^{\text{univ}} \otimes_{\mathcal{O}} \mathcal{O}[\Delta/(c+1)] \).

**Proof.** Let \( r_S,\psi_0 : G_{F^+,S} \to \mathcal{G}_n(R_{S,\psi_0}^{\text{univ}}) \) denote a representative of the universal deformation, and let \( \Psi : G_{F,S} \to \mathcal{O}[\Delta/(c+1)]^\times \) denote the universal character. The map
\[
R_S^{\text{univ}} \to R_{S,\psi_0}^{\text{univ}} \otimes_{\mathcal{O}} \mathcal{O}[\Delta/(c+1)]
\]
is induced by the lifting \( r_S,\psi_0 \otimes \Psi \), which is of type \( S \). We construct its inverse. Indeed, let \( \psi_S = \det r_S|_{G_{F,S}} \).

Since \( l \) does not divide \( n \), there is a unique character \( \psi : \Delta/(c+1) \to (R_S^{\text{univ}})^\times \) such that \( \psi^n = \psi_S \psi_0^{-1} \). The lifting \( r_S \otimes \psi \) has determinant \( \Psi \), and the pair \( (r_S \otimes \psi^{-1}, \psi) \) induces a homomorphism
\[
R_{S,\psi_0}^{\text{univ}} \otimes_{\mathcal{O}} \mathcal{O}[\Delta/(c+1)] \to R_S^{\text{univ}}
\]
which is the inverse of the isomorphism of the lemma. \( \square \)

### 3.7 Localizing at a dimension one prime

In this section we assume that \( r \) is Schur and that \( \tau|_{G_{F,S}} = \bar{p}_1 \oplus \bar{p}_2 \) is a direct sum of two absolutely irreducible representations, and that \( n \) is coprime to \( l \). We will later have need to consider the following situation. Let \( p \subset R_S^{\text{univ}} \) be a dimension one prime of characteristic \( l \), and let \( q = p \cap P_S \). Write \( A \) for the normalization of \( R_S^{\text{univ}}/p \) in its residue field \( F \), and \( E' \) for the residue field of \( P_S/q \). Thus \( A \cong k'[\mathbb{T}] \). After possibly enlarging \( \mathcal{O} \), we can suppose that \( k = k' \), and we will assume this in what follows. We will also assume that \( |F^+: \mathbb{Q}| > 1 \).

We suppose that \( \Lambda \to A \) is finite; in this case, we can choose a finite faithfully flat extension \( \Lambda \to \Lambda \) inducing a bijection on minimal primes, together with a surjective map \( \Lambda \to A \) with kernel \( \overline{P} \) making the diagram
\[
\Lambda \xrightarrow{\phi} \Lambda \xrightarrow{\rho} A
\]
commute. We can further suppose that for each minimal prime \( Q \subset \Lambda \), \( \Lambda/Q \) is isomorphic to a power series ring over \( \mathcal{O} \), and that the map \( \Lambda/(Q, \lambda) \to \Lambda/(Q, \lambda) \) induces a separable extension of fraction fields. Indeed, we can find an isomorphism \( \Lambda \cong \mathcal{O}[X_1, \ldots, X_n] \otimes_{\mathcal{O}} \mathcal{O}[\Delta] \), where \( \Delta \) is a finite abelian \( l \)-group. The homomorphism \( \Lambda \to A \cong k'[\mathbb{T}] \) is given on the first factor by \( X_i \mapsto T^{u_i} u_i \), where each \( u_i \in A \) is either zero or a unit, and on the second factor by the augmentation homomorphism \( \mathcal{O}[\Delta] \to k \). We may suppose that \( u_1 \) is a unit. We set \( \Lambda = \mathcal{O}[W_1, \ldots, W_s] \otimes_{\mathcal{O}} \mathcal{O}[\Delta] \). The homomorphism \( \Lambda \to A \) is given by \( W_i \mapsto W_i, W_i \mapsto 0, i = 2, \ldots, s \). For \( i = 1, \ldots, s \), let \( \bar{u}_i \) denote an arbitrary lift of \( u_i \) to \( \mathcal{O}[W_1] \subset \Lambda \). Then the homomorphism \( \Lambda \to \Lambda \) is given by the formulae \( X_i \mapsto W_1^{u_1} u_i + W_2 W_1^{-1} u_i, X_i \mapsto -W_i + W_1^{u_1} u_i, i = 2, \ldots, s \).

Let \( r_S \) be a representative of the universal deformation. Then the homomorphism \( R_S^{\text{univ}} \to A \) induces a representation
\[
r_S \mod p : G_{F^+,S} \to \mathcal{G}_n(A),
\]
which we will denote by \( r_p \). Suppose that \( r_p|_{G_{F,S}} \otimes_A E \) is absolutely irreducible. Then the group \( \mu_2 \times \mu_2 \), acting on \( R_S^{\text{univ}} \), permutes transitively the set of primes of \( R_S^{\text{univ}} \) above \( q \). We can define primes \( \bar{p} = \ker(R_S^{\text{univ}} \otimes_A \Lambda \to A) \) and \( \bar{q} = \ker(P_S \otimes_A \Lambda \to A) \). The group \( \mu_2 \times \mu_2 \) acts on \( R_S^{\text{univ}} \otimes_A \Lambda \), acting trivially on the second factor.

**Proposition 3.33.** Suppose that \( r_p|_{G_{F,S}} \otimes_A E \) is absolutely irreducible.

1. Fix an integer \( q \). Then for any choice of Taylor-Wiles data of order \( q \) and level \( N \), we have an auxiliary deformation problem \( S_N \), and primes \( p_N, q_N, \bar{p}_N, \bar{q}_N \) of \( R_S^{\text{univ}}, P_S, R_S^{\text{univ}} \otimes_A \Lambda \) and \( P_S \otimes_A \Lambda \), respectively. There exists an integer \( C > 0 \) depending only on \( S, q \) and \( p \) such that for any choice of Taylor-Wiles data as above, the \( A \)-module \( \bar{p}_N/(\bar{q}_N + \bar{p}_N^2) \) is finite, of order at most \( C \).
ii) Suppose that $E = E'$. Then the primes of $R^\text{univ}_S \otimes \Lambda \tilde{\Lambda}$ above $\tilde{q}$ are in bijection with the primes of $R^\text{univ}_S$ above $q$. The natural map on localizations and completions $(P_S \otimes \Lambda \tilde{\Lambda})_{\tilde{q}} \to (R^\text{univ}_S \otimes \Lambda \tilde{\Lambda})_{\tilde{p}}$ is an isomorphism.

\textbf{Proof.} Elements in the $A$-Pontryagin dual of $\tilde{b}/(\tilde{q} + \tilde{p}^2)$ correspond to equivalence classes of lifts $\tilde{r}$ of $r_p$ to $A \oplus \epsilon E/A$ with characteristic polynomial equal to that of $r_p$, subject to certain ramification conditions at places in $S$. We can write $\tilde{r} = (1 + \phi)r_p$, where $\phi \in Z^1(G_{F^+, S}, ad r_p \otimes A E/A)$. It therefore suffices to show the following two claims: first, that the minimal number of generators of the finitely generated $A$-module $H^1(G_{F^+, S}, ad r_p \otimes A E/A)$ can be bounded solely in terms of $|S' - S|$; second, that there exists an $N_0$, depending only on the image of $r_p$, such that for any such $\phi$, $T^{N_0} \phi$ is a coboundary.

For the first claim, it suffices since $\pi$ is Schur to bound $\dim_k H^1(G_{F, S'}, ad \pi)$ solely in terms of $|S' - S|$, and this is immediate. For the second, it suffices even to show that we can find $N_0$ such that the restriction of $T^{N_0} \phi$ to $Z^1(G_{F, S}, ad r_p \otimes E/A)$ is a coboundary. Extend the representation $r_p|_{G_{F, S}}$ to an algebra map $r_p|_{G_{F, S}} : A[G_{F, S}] \to M_{n \times n}(A)$. Since $r_p|_{G_{F, S}} \otimes A E$ is absolutely irreducible, this algebra homomorphism is surjective after extending scalars to $E$, and so $A = r_p|_{G_{F, S}}(A[G_{F, S}])$ is an order in $M_{n \times n}(A)$. Choose $N \geq 0$ such that $T^N M_{n \times n}(A) \subset A$.

Writing $\tilde{r}(a) = r_p(a) + \delta(a)$, we have the formula $\delta(ab) = r_p(a)\delta(b) + \delta(a)r_p(b)$, and $tr \delta(a) \in E/A$ is zero for all $a \in A[G_{F, S}]$. Suppose that $a$ belongs to the ideal $r_p|_{G_{F, S}} \subset A[G_{F, S}]$. Then $\delta(ab) = \delta(a)r_p(b)$. We deduce that $tr T^N \delta(a) M_{n \times n}(A) = 0$, hence $T^N \delta(a) = 0$. Replacing $\delta$ with $T^N \delta$, we may assume that $\delta$ is pulled back from an $A$-linear derivation $A \to M_{n \times n}(E/A)$.

Let $B = \sum_{j=1}^n \delta(T^N e_{j,1} T^N e_{1,j})$, where $e_{j,1}$ is the matrix with a 1 in the $(j, i)$ spot and 0 everywhere else. For any $\gamma \in A$, we have

$$T^{2N} \delta(\gamma) = \delta(T^{2N} \gamma) = \sum_{j=1}^n \delta(T^N e_{j,1} T^{2N} e_{1,j} \gamma) = \sum_{j=1}^n \delta(T^N e_{j,1} T^{N} e_{1,j} \gamma) + \sum_{j=1}^n \delta(T^{2N} e_{j,1} T^N e_{1,j} \gamma),$$

and this last term equals $B\gamma + \sum_{j=1}^n T^N e_{j,1} \delta(T^{N} e_{1,j} \gamma)$. On the other hand, we have

$$\sum_{j=1}^n T^N e_{j,1} \delta(T^{N} e_{1,j} \gamma) = \sum_{j=1}^n T^N e_{j,1} \left( \sum_{i=1}^n \gamma_{j,i} T^{N} e_{1,i} \right) = \sum_{j=1}^n \sum_{i=1}^n \gamma_{j,i} T^{N} e_{j,1} \delta(T^{N} e_{1,i}),$$

and this equals $-\gamma B$. It follows that after multiplying $\delta$ by $T^{2N}$, it becomes a coboundary. Thus we can take $N_0 = 3N$.

For the second part, note that the fiber of $\text{Spec } R^\text{univ}_S \otimes \Lambda \tilde{\Lambda} \to \text{Spec } P_S \otimes \Lambda \tilde{\Lambda}$ above $\tilde{q}$ is identified with $\text{Spec } R^\text{univ}_S \otimes P_S E$, thus has exactly two points which are permuted by the action of the group $\mu_2 \times \mu_2$. The map

$$(P_S \otimes \Lambda \tilde{\Lambda})_{\tilde{q}} \to (R^\text{univ}_S \otimes \Lambda \tilde{\Lambda})_{\tilde{p}}$$

is surjective, since it is so on relative tangent spaces. For the injectivity, note that for any prime $\tilde{r}$ of $R^\text{univ}_S \otimes \Lambda \tilde{\Lambda}$ above $q$, the completions $(R^\text{univ}_S \otimes \Lambda \tilde{\Lambda})_{\tilde{r}}$ and $(R^\text{univ}_S \otimes \Lambda \tilde{\Lambda})_{\tilde{p}}$ are isomorphic as $(P_S \otimes \Lambda \tilde{\Lambda})_{\tilde{q}}$-algebras, via the action of the group $\mu_2 \times \mu_2$. The map

$$(P_S \otimes \Lambda \tilde{\Lambda})_{\tilde{q}} \to R^\text{univ}_S \otimes P_S (P_S \otimes \Lambda \tilde{\Lambda})_{\tilde{q}} \cong \prod_{\tilde{r}} (R^\text{univ}_S \otimes \Lambda \tilde{\Lambda})_{\tilde{r}}$$

is injective, since $(P_S \otimes \Lambda \tilde{\Lambda})_{\tilde{q}}$ is a flat $P_S$-algebra. It follows that the induced map $(P_S \otimes \Lambda \tilde{\Lambda})_{\tilde{q}} \to (R^\text{univ}_S \otimes \Lambda \tilde{\Lambda})_{\tilde{p}}$ must also be injective.

It need not always be the case that $E' = E$. However, let $S$ denote the deformation problem of the previous section. Let $\psi : \Delta/(c + 1) \to A^c$ denote a continuous character which has trivial reduction modulo $T$. Then we have a lifting $r_p \otimes \psi : G_{F^+, S} \to G_0(A)$, and we write $p_{\psi}$ and $q_{\psi}$ for the induced ideals of $R^\text{univ}_S$ and $P_S$, respectively.

\textbf{Lemma 3.34.} (i) Let $Q$ be a minimal prime of $R^\text{univ}_S$. Then $Q \subset p$ if and only if $Q \subset p_{\psi}$.\hfill $\square$
(ii) There exists a choice of \( \psi \) with the property that \( \text{Frac } P_S/q_\psi = \text{Frac } R_S^{\text{univ}}/p_\psi = E \).

Proof. For the first part, it suffices by symmetry to show that if \( Q \subset p \) then \( Q \subset p_\psi \). Let \( r \subset p \) be a prime minimal over \((Q, \lambda)\). We show that \( r \subset p_\psi \) also. There is an isomorphism \( R_S^{\text{univ}}/\lambda \cong R_S^{\text{univ}}/\lambda \otimes_k k[\Delta] \), and \( \text{Spec } k[\Delta] \) is geometrically irreducible. Since the homomorphisms \( R_S^{\text{univ}} \rightarrow A \) induced by the liftings \( r_p \) and \( r_p \otimes \psi \) are the same, the result follows.

For the second part, choose a surjection \( \Delta/(c + 1) \rightarrow \mathbb{Z}_l \), and \( \sigma \in G_{F,S} \) that is mapped to a topological generator of \( \mathbb{Z}_l \). Write \( \det r_{\sigma} |_{G_{F,S}}(\sigma) = x\alpha \), where \( x \in k^\times \) and \( \alpha \in A^\times \) satisfies \( \alpha \equiv 1 \mod T \). Define a character \( \psi : \Delta/(c + 1) \rightarrow A^\times \) by pulling back the character \( \mathbb{Z}_l \rightarrow A^\times \) which sends the image of \( \sigma \) to the unique \( n^{\text{th}} \) root of \((1 + T)/\alpha \) in \( 1 + TA \). Then \( \det (r \otimes \psi) |_{G_{F,S}}(\sigma) = x(1 + T) \), and \( P_S/q_\psi \) identifies with a closed subfield of \( E = k((T)) \) containing \( T \). The only possibility is \( P_S/q_\psi = E \), and this implies the lemma.

3.8 Connectedness dimension

Let \( R \) be a complete local Noetherian \( \mathcal{O} \)-algebra. By definition the connectedness dimension of \( R \) is

\[
c(R) = \inf_{C_1, C_2} \{ \dim \cup_{C \in C_1, D \in C_2} C \cap D \},
\]

where the infimum is taken over the set of partitions of the set of irreducible components of \( \text{Spec } R \) into two disjoint non-empty subsets \( C_1, C_2 \).

If \( I \subset R \) is an ideal, the arithmetic rank \( r(I) \) of \( I \) is the minimal integer \( r \) such that there exist elements \( f_1, \ldots, f_r \) with \( \sqrt{(f_1, \ldots, f_r)} = \sqrt{I} \). The following result follows immediately from [BR86, Theorem 2.4].

Proposition 3.35. With \( R, I \) as above, let \( S = R/I \). Then we have \( c(S) \geq c(R) - r(I) - 1 \).

4 Automorphic forms

In this section we define the spaces of algebraic modular forms on definite unitary groups that we will use. We work with ordinary forms; by Hida theory, it suffices therefore to work throughout with forms of weight zero (or equivalently, with trivial coefficients). A development of this theory in the context of Galois representations was given by Geraghty [Ger]. We will refer mostly to this work for structural results. We remark that Geraghty works with ‘true’ unitary groups (associated to a Hermitian vector space), whereas we prefer to work with a division algebra endowed with involution at the second kind, which may therefore be ramified at a finite set of places (the set \( S(B) \) below). This does not, however, require any modifications to the Hida-theoretic arguments, for which we continue to use Geraghty’s work as a reference.

4.1 Definitions

Fix an integer \( n \geq 1 \) and an odd prime \( l \). Let \( L/L^+ \) be CM imaginary extension of a totally real field. We suppose that every prime above \( l \) splits in \( L/L^+ \) and that \( L/L^+ \) is unramified. Let \( S_l \) denote the set of places of \( L^+ \) dividing \( l \). For each place in \( v \in S_l \) we choose a place \( \overline{v} \) of \( L \) above it and denote the set of these by \( \overline{S_l} \). We write \( \overline{I}_l \) for the set of embeddings \( L \hookrightarrow \mathbb{Q}_l \), and \( \overline{I}_l \subset I_l \) for those embeddings inducing an element of \( \overline{S}_l \). We fix also a finite non-empty set of places \( S(B) \) of \( L^+ \) such that

- Every element of \( S(B) \) splits in \( L \).
- \( S(B) \) contains no place above \( l \).
- If \( a \) is even then \( a|L^+:\mathbb{Q}|/2 + |S(B)| \equiv 0 \mod 2 \).

Under these hypotheses we can find a central division algebra \( B \) over \( L \) equipped with an involution \( \dagger \) satisfying the following properties (see [CHT08, §3.3]):
• \(\dim L = n^2\).
• \(B^{op} \cong B \otimes_{L,v} L\).
• \(B\) splits outside \(S(B)\).
• For each prime \(w\) of \(L\) above \(S(B)\), \(B_w\) is a division algebra.
• \(\mathfrak{m}_L = c\).
• Defining a group \(G\) over \(L^+\) by the formula

\[
G(R) = \{g \in B \otimes_{L^+} R \mid g^1 g = 1\}
\]

for any \(L^+\)-algebra \(R\), we have that \(G(L^+ \otimes_{\mathbb{Q}} \mathbb{R})\) is compact and \(G\) is quasi-split at every finite place \(v \notin S(B)\) of \(L^+\).

We can find a maximal order \(O_B \subset B\) such that \(O_B^1 = O_B\) and such that \(O_{B,w}\) is a maximal order in \(B_w\) for every place \(w \in L\) split over \(L^+\). This defines an integral model for \(G\) over \(O_{L^+}\), which we continue to denote by \(G\).

Let \(v \notin S(B)\) be a finite place of \(L^+\) which splits as \(v = w\mathfrak{m}_v\) in \(L\). Then we can find an isomorphism

\[
\iota_v : O_{B,v} \to M_n(O_{L,v}) \cong M_n(O_{L_w}) \times M_n(O_{L_w})
\]

such that \(\iota_v(g^1) = \iota_v(g)^c\). Projection to the first factor then gives rise to an isomorphism

\[
\iota_w : G(L^+_w) \to GL_n(L_w),
\]

with the property that \(\iota_w(G(O_{L^+_w})) = GL_n(O_{L,v})\). If \(v \in S(B)\) then we get an isomorphism \(\iota_w : G(L^+_v) \to B^\times_w\), with the property that \(\iota_w(G(O_{L^+_v})) = O_{B,v}^\times\).

Let \(K\) be a finite extension of \(\mathbb{Q}_l\) in \(\overline{\mathbb{Q}}_l\), with ring of integers \(O\) and residue field \(k\). We write \(\lambda\) for the maximal ideal of \(O\). We will suppose \(K\) large enough to contain the image of every embedding of \(L\) in \(\overline{\mathbb{Q}}_l\).

Let \(R\) be a finite set of finite places of \(L^+\), disjoint from \(S_l \cup S(B)\) and containing only places which split in \(L\). Let \(T \supset S_l \cup S(B) \cup R\) be a finite set of places of \(L^+\) which split in \(L\). For each \(v \in T\) we choose a place \(\mathfrak{p}\) of \(L\) above it, extending our previous choice for \(v \in S_l\).

We suppose that \(U = \prod_v U_v\) is an open compact subgroup of \(G(A_{L^+}^\infty)\) such that \(U_v \subset \iota_v^{-1}(Iw(\mathfrak{p}))\) for \(v \in R\) and \(U_v = \iota_v^{-1}(O_{B,v}^\times)\) for \(v \in S(B)\). Here given a place \(\mathfrak{p}\) of \(L\), \(Iw(\mathfrak{p})\) is the subgroup of \(GL_n(O_{L,v})\) consisting of matrices whose image in \(GL_n(k(\mathfrak{p}))\) is upper triangular. We will also write \(Iw_1(\mathfrak{p}) \subset Iw(\mathfrak{p})\) for the subgroup of matrices whose image in \(GL_n(k(\mathfrak{p}))\) is upper-triangular and unipotent. We say that \(U\) is sufficiently small if there exists a place \(v \notin S_l\) of \(L^+\) such that \(U_v\) contains no non-trivial elements of finite order. We shall often assume this in the applications below.

For each \(v \in R\), we choose a character \(\chi_v : Iw(\mathfrak{p})/Iw_1(\mathfrak{p}) \to O^\times\). We write \(M = \otimes_{v \in R} O(\chi_v)\). This is equivalent to the data of an \(n\)-tuple of characters \(\chi_v, \ldots, \chi_v,n : k(\mathfrak{p})^\times \to O^\times\).

**Definition 4.1.** Let \(U, \chi = \{\chi_v\}\) be as above. If \(A\) is an \(O\)-module, we write \(S_{\chi}(U, A)\) for the set of functions

\[
f : G(L^+) \setminus G(A_{L^+}^\infty) \to M \otimes O A
\]

such that for every \(u \in U\), we have \(f(gu) = u_R^{-1} f(g)\), where \(u_R\) denotes the image of \(u\) under the projection \(U \to \prod_{v \in R} Iw(\mathfrak{p})\). If \(V\) is an arbitrary subgroup of \(G(A_{L^+}^\infty)\) whose projection to the places \(v \in R\) is contained in \(\iota_v^{-1}(Iw(\mathfrak{p}))\), then we will write

\[
S_{\chi}(V, \overline{\mathbb{Q}}_l) = \lim_{U \supset V} S_{\chi}(U, \overline{\mathbb{Q}}_l),
\]

the limit being taken over all open compact subgroups \(U\) as above.
For integers $0 \leq b \leq c$, and $v \in S_l$, we consider the subgroup $Iw(\varpi^{b,c}) \subset GL_n(O_{L_v})$ defined as those matrices which are congruent to an upper-triangular matrix modulo $\varpi^b$ and congruent to a unipotent upper-triangular matrix modulo $\varpi^c$. We set $U^{(b,c)} = U^l \times \prod_{v \in S_l} T^{-1}_v Iw(\varpi^{b,c})$.

We now define the Hecke operators that we will be using. If $w$ is a place of $L$ split over $L^+$ and not in $T$, let $\varpi_w$ be a uniformizer of $L_w$. Then we define for $j = 1, \ldots, n$ the Hecke operator

$$T^j_w = t_w^{-1}(\left[GL_n(O_{L_w}) \left( \begin{array}{cc} \varpi_w^{-1} & 0 \\ 0 & 1_{n-j} \end{array} \right) GL_n(O_{L_w}) \right]).$$

This is independent of the choice of uniformizer $\varpi_w$.

If $v$ is a place of $L^+$ dividing $l$ and if $u \in T_n(O_{L_v})$, where $T_n \subset GL_n$ is the diagonal torus, then we write

$$\langle u \rangle = t_w^{-1}(\left[Iw(\varpi^{b,c})u \right] Iw(\varpi^{b,c})).$$

If $\varpi_v$ is a uniformizer of $L_v$, we define

$$U^j_w \varpi_v = t_w^{-1}(\left[Iw(\varpi^{b,c}) \left( \begin{array}{cc} \varpi_v^{-1} & 0 \\ 0 & 1_{n-j} \end{array} \right) Iw(\varpi^{b,c}) \right]).$$

Now [Ger, Lemma 2.3.3] shows that these operators act on the spaces $S^\chi(U^{(b,c)}), A)$ for any $O$-module $A$ and commute with the inclusions

$$S^\chi(U^{(b,c)}), A) \subset S^\chi(U^{(b',c')}, A),$$

when $b \leq b'$ and $c \leq c'$.

For any $O$-module $A$ there is defined a subspace of ordinary forms

$$S^{\text{ord}}_{\chi}(U^{(b,c)}), A) = cS^\chi(U^{(b,c)}), A) \subset S^\chi(U^{(b,c)}), A)$$

which is preserved by all of the above defined Hecke operators, defined as the image of the ordinary projector $e$, see [Ger, Definition 2.4.2].

**Definition 4.2.** We write $T^T\chi(U^{(b,c)}), A)$ for the $O$-subalgebra of

$$\text{End}_O(S^\text{ord}_{\chi}(U^{(b,c)}), A))$$

generated by the operators $T^j_w$, $j = 1, \ldots, n$ and $(T^n_w)^{-1}$ as above and $\langle u \rangle$ for

$$u \in T_n(O_{L^+,l}) = \prod_{v \in S_l} T_n(O_{L_v}).$$

We write $T^T\chi(U^{(\infty)}), O) = \lim_{\longrightarrow c} T^T\chi(U^{(\infty)}, O)$. This algebra acts naturally on the spaces

$$S^\chi(U^{(\infty)}, K/O) = \lim_{\longrightarrow c} S^\chi_{\text{ord}}(U^{(\infty)}, K/O)$$

and

$$S^\chi(U^{(\infty)}, O) = (S^\chi(U^{(\infty)}, K/O))^\vee$$

where $(\cdot)^\vee$ denotes $\text{Hom}_O(\cdot, K/O)$.

We write $T_n(l)$ for the maximal pro-$l$ subgroup of $T_n(O_{L^+,l})$. Define a homomorphism $T_n(l) \to T^T\chi(U^{(\infty)}, O)^\times$ by the formula

$$(u_v)_{v \in S_l} \mapsto \prod_{v \in S_l} \langle u_v \rangle \cdot \prod_{i=1}^n \tau(u_{\varpi}(\tau, i))^{1-i},$$
where the product is over embeddings $\tau : L \to K$ inducing a place of $\mathcal{B}$ and $\overline{v}(\tau)$ denotes the place induced by such an embedding, and we write $u = (u_1, \ldots, u_n)$ for a typical element of $T_n$. This choice defines a homomorphism (after composing with the Artin map)

$$\Lambda \to \mathbb{T}_\chi^T(U(\mathbb{R}^\infty), \mathcal{O}),$$

where $\Lambda$ denotes the completed group ring of $I_{K_0}(l)^n$, and we will always view the Hecke algebra $\mathbb{T}_\chi^T(U(\mathbb{R}^\infty), \mathcal{O})$ as endowed with this $\Lambda$-algebra structure.

**Proposition 4.3.** Suppose that $U$ is sufficiently small. Then $\mathbb{T}_\chi^T(U(\mathbb{R}^\infty), \mathcal{O})$ is a finite torsion-free $\Lambda$-algebra and $S_\chi(U(\mathbb{R}^\infty), \mathcal{O})$ is a faithful $\mathbb{T}_\chi^T(U(\mathbb{R}^\infty), \mathcal{O})$-module and finite free over $\Lambda$ with the induced $\Lambda$-module structure.

**Proof.** This is proved exactly as in [Ger] Proposition 2.5.3 and [Ger] Corollary 2.5.4.

### 4.2 Galois representations

**Theorem 4.4.** Let $\pi$ be an irreducible $G(k_{L_{\mathbb{R}^\infty}}) \times \prod_{v \in R} \mathbb{T}_\chi^T(U(\mathbb{R}^\infty), \mathcal{O})$-constituent of the space of automorphic forms $S_\chi(\prod_{v \in R} \mathbb{T}_\chi^T(U(\mathbb{R}^\infty), \mathcal{O}))$. Then there exists a continuous semisimple representation

$$r_\ell(\pi) : G_L \to \text{GL}_n(\overline{\mathbb{Q}}_l),$$

unique up to isomorphism, such that:

(i) If $v \notin S_l \cup S(B)$ is a place of $L^+$ which splits as $v = wv'$ in $L$, then

$$(r_\ell(\pi)|_{G_{L_{wv'}}})^{ss} \cong (r_\ell(\pi_v \circ i^{-1}_w))^{(1 - n)}.$$ 

(ii) $r_\ell(\pi)_v = r_\ell(\pi)_v^\nu e^{1-n}$.

(iii) If $v$ is an inert place and $\pi_v$ has a fixed vector for a hyperspecial maximal compact in $G(L_v^+)$ then $r_\ell(\pi)$ is unramified above $v$.

(iv) If $v \in R$ and $\pi_v \circ i^{-1}_w \neq 0$, then for every $\sigma \in I_{L_v}$, we have

$$\text{char}_{r_\ell(\pi)(\sigma)}(X) = \prod_{j=1}^{n} (X - \chi_{w,j}^{-1} \circ \text{Art}_{L_v}(\sigma)).$$

(v) If $v \in S_l$ splits as $v = wv'$ then $r_\ell(\pi)|_{G_{F_w}}$ is de Rham. If $\pi_w$ is unramified then $r_\ell(\pi)|_{G_{L_w}}$ is even crystalline. Moreover, for each $\tau \in I_l$, we have

$$\text{HT}_\tau(r_\ell(\pi)) = \{0, 1, \ldots, n - 1\}.$$ 

Finally if $r_\ell(\pi)$ is irreducible then for each place $v$ of $L^+$ which splits as $L$ as $ww'$, the representation $\pi_w \circ i^{-1}_w$ is generic.

**Proof.** This follows from [CHT08] Proposition 3.3.4].

**Proposition 4.5.** Suppose that $U$ is sufficiently small. Let $m$ be a maximal ideal of $\mathbb{T}_\chi^T(U(\mathbb{R}^\infty), \mathcal{O})$. Then, after possibly extending $\mathcal{O}$, we can identify $\mathbb{T}_\chi^T(U(\mathbb{R}^\infty), \mathcal{O})/m = k$ and there is a continuous semisimple representation

$$r_m : G_L \to \text{GL}_n(k),$$

unique up to isomorphism, such that:

(i) $r_m^{\nu} \cong r_m^{\nu} e^{1-n}$. 

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(ii) $\mathfrak{m}$ is unramified outside $T$. For all $v \not\in T$ splitting in $L$ as $v = ww^c$, the characteristic polynomial of $\mathfrak{m}(\text{Frob}_w)$ is
\[ X^n + \cdots + (-1)^i (q_w)^{j-i-1}/2T_w^iX^{n-j} + \cdots + (-1)^n (q_w)^{n(n-1)/2}T_w^n. \]

Proof. This is proved exactly as in [Ger, Proposition 2.7.3]. We remark that the proof shows that any maximal ideal $\mathfrak{n} \subset T^T(U(I^\infty), \mathcal{O})$ is pulled back via the map $T^T(U(I^\infty), \mathcal{O}) \to T^T(U(I^{1/2}), \mathcal{O})$.\hfill \Box

Definition 4.6. We say that a maximal ideal $\mathfrak{m}$ as above is residually Schur if writing $\mathfrak{m} = \bigoplus_{i=1}^{\epsilon} \mathfrak{p}_i$ as a sum of irreducible subrepresentations, each $\mathfrak{p}_i$ is absolutely irreducible and for each $1 \leq i, j \leq s$, we have $\mathfrak{p}_i \not\sim \mathfrak{p}_j$.

To $\pi$ as above, we can associate a maximal ideal $\mathfrak{m}$. Indeed, the Hecke eigenvalues of $\pi$ give rise to a homomorphism $T^T(V(U(I^\infty)), \mathcal{O}) \to \mathcal{Z}_l$, and we take $\mathfrak{m}$ to be the kernel of this homomorphism. If this ideal is residually Schur then it is easy to see using [CHT08, Proposition 3.3.2] that the weak base change to $\mathcal{O}(\mathbb{A}_L)$ is cuspidal, and $r_1(\pi)$ is irreducible, by [TY07, Corollary B].

We fix from now on a residually Schur maximal ideal $\mathfrak{m}$.

Proposition 4.7. Suppose that $\mathfrak{m}$ is residually Schur. Then, after possibly enlarging $K$, $\mathfrak{m}$ admits an extension to a continuous homomorphism
\[ \tilde{\mathfrak{m}} : G_{L^+} \to \mathcal{G}_n(k), \]
with the property that $\tilde{\mathfrak{m}}^{-1}(\mathbb{G}_m \times \mathbb{G}_l(k)) = G_L$ and $\nu \circ \tilde{\mathfrak{m}} = e^{1-n}\delta_{L^+/L}$. Moreover, this extension is Schur.

Proof. Choose an irreducible constituent $\pi$ of $S_{\chi}(\prod_{v \in R} \ell_v^{-1}(\mathcal{Q}_v))$ such that $r_1(\pi)$ has reduction equal to $\mathfrak{m}$. By [CHT08, Lemma 2.1.5], after possibly extending $K$ we can assume that $r_1(\pi)$ is valued in $\mathbb{G}_n(\mathcal{O})$ and find an extension to a representation $r_1(\pi) : G_{F+c} \to \mathcal{G}_n(\mathcal{O})$. By the main result of [BCT], we have $\nu \circ r_1(\pi) = e^{1-n}\delta_{L^+/L}$. It now suffices to take the reduction mod $\lambda$ of this extension. The representation obtained in this way is Schur, by Lemma 3.4.\hfill \Box

Proposition 4.8. Suppose that $U$ is sufficiently small and that $\mathfrak{m}$ is residually Schur. Fix $c \geq 1$, and let $\mathfrak{m}_c \subset T^T(U(I^c), \mathcal{O})$ be pushforward of $\mathfrak{m}$, a maximal ideal (cf. the proof of Proposition 3.22). Let $\mathfrak{p} \supseteq \mathfrak{m}_c$ be a minimal prime. Then there exists a finite field extension $E_p$ of the fraction field of $T^T(U(I^c), \mathcal{O})/\mathfrak{p}$ with ring of integers $\mathcal{O}_p$ and a continuous representation $r_p : G_{L^+, c} \to \mathcal{G}_n(\mathcal{O}_p)$ such that $r_p$ mod $\mathfrak{m}_c = \tilde{\mathfrak{m}}$ and satisfying the following: for any embedding $E_p \hookrightarrow \mathcal{Q}_l$, there is an irreducible $G(\mathbb{A}_{L^+, c}^\infty \times \prod_{v \in R} \ell_v^{-1}(\mathcal{Q}_v))$-constituent of the space $S_{\chi}(\prod_{v \in R} \ell_v^{-1}(\mathcal{Q}_v))$ inducing the homomorphism $T^T(U(I^c), \mathcal{O}) \to \mathcal{Q}_l$, and such that $r_p_{|G_{L^+, c} \otimes \mathcal{O}_p} \mathcal{Q}_l \cong r_1(\pi)$.\hfill \Box

Proof. This follows the existence of $r_1(\pi)$ and [CHT08, Lemma 2.1.5].

In contrast to the residually irreducible case, $\mathfrak{m}$ need not admit a lifting to $T^T(U(I^\infty), \mathcal{O})$. The best we can hope for is the following.

Proposition 4.9. Suppose that $U$ is sufficiently small and that $\mathfrak{m}$ is residually Schur. Define a group determinant over $k$ by $D_{\mathfrak{m}} = \det \circ \tau_{\mathfrak{m}}|_{G_{L,S}}$. Then $D_{\mathfrak{m}}$ admits a unique lifting to a pseudodeformation
\[ D_{\mathfrak{m}} : G_{L,S} \to T^T(U(I^\infty), \mathcal{O})_{\mathfrak{m}} \]
satisfying the following property: for all $v \not\in T$ splitting in $L$ as $v = ww^c$, the characteristic polynomial of $D_{\mathfrak{m}}(\text{Frob}_w)$ is
\[ X^n + \cdots + (-1)^i (q_w)^{(j-i)/2}T_w^iX^{n-j} + \cdots + (-1)^n (q_w)^{n(n-1)/2}T_w^n. \]

Proof. It suffices to construct such a lifting to $D_{\mathfrak{m}}(c) : G_L \to T^T(U(I^c), \mathcal{O})_{\mathfrak{m}_c}$, as the uniqueness ensures that we can then pass to the limit. Let $A \subset k \subset \bigoplus_{\mathfrak{p}} \mathcal{O}_p$ denote the subring of elements $(x, y_p)$ such that $x \equiv y_p$ mod $\mathfrak{m}_c$ for each $\mathfrak{p}$. By the previous proposition, there exists a continuous representation $r(c) = \bigoplus_{\mathfrak{p}} r_p : G_{L^+, S} \to \mathcal{G}_n(A)$, and we now take $D_{\mathfrak{m}}(c) = \det \circ r(c)|_{G_{L,S}}$. By the second part of Proposition 3.22, $D_{\mathfrak{m}}(c)$ is actually valued in $T^T(U(I^c), \mathcal{O})_{\mathfrak{m}_c} \subset A$. It is now easy to see that $D_{\mathfrak{m}}(c)$ satisfies the required properties.\hfill \Box
4.3 Deformation rings and Hecke algebras

We now specialize to the following situation. Suppose that \( T = S = S_f R \cup S(B) \cup S_a \), and that \( U = \prod_v U_v \) has the following form:

- For \( v \) inert in \( L \), \( U_v \subset G(L_v^+) \) is a hyperspecial maximal compact subgroup.
- For \( v \notin T \) split in \( L \), \( U_v = G(O_{L_v^+}) \).
- For \( v \in S(B), U_v \) is a maximal compact subgroup.
- For \( v \in S_t, U_v = G(O_{L_v^t}) \).
- For \( v \in R, U_v = \iota_v^{-1} \text{Iw}(\bar{v}) \).
- For \( v \in S_a, U_v = \iota_v^{-1} \ker(\text{GL}_n(O_{L_v^t}) \to \text{GL}_n(k(\bar{v}))) \).

We suppose that \( S_a \) is non-empty and that for every \( v \in S_a \), \( v \) is absolutely unramified, \( \tau_m \) is unramified above \( v \), \( \text{ad}(\pi_{\text{Frob}_v}) = 1 \) and \( v \) does not split in \( F(\zeta_l) \). Then \( H^0(G_{L_v^t}, \text{ad}\pi(1)) = 0 \), and \( U \) is sufficiently small. We leave the characters \( \gamma_v \) for \( v \in R \) unspecified, but suppose that they are trivial mod \( \lambda \). We suppose that \( m \subset T^T(U(\infty), O) \) is a residually Schur maximal ideal.

Suppose also that for each \( v \in S_t R \cup S(B) \), \( \tau_m |_{G_{L_v^t}} \) is trivial and that for each \( v \in R \cup S(B) \), \( q_v \equiv 1 \mod l \). Under these assumptions we can define a global deformation problem (using the local deformation problems defined in [4]):

\[
S_\chi = \left( L/L^+, T, \tilde{T}, \Lambda, \tau_m, \infty, \delta^\Lambda_{L^+/L^+}, \{ R_v^\Lambda \}_{v \in S_t} \cup \{ R_v^\infty \}_{v \in R} \cup \{ R_v^{St} \}_{v \in S(B)} \cup \{ R_v^\square \}_{v \in S_a} \right).
\]

Proposition 4.10. The natural map \( Q_S \hat{\otimes}_O \Lambda \to T^T(U(\infty), O)_m \) is surjective and factors through \( P_S \subset R^{\text{univ}}_S \).

Proof. The map is surjective since since \( T^T(U(\infty), O)_m \) is topologically generated as a \( \Lambda \)-algebra by the coefficients of the characteristic polynomial of the group determinant \( D_m \), evaluated at the Frobenius elements at places of \( L \) split over \( L^+ \). Let \( I = \ker(Q_S \hat{\otimes}_O \Lambda \to R^{\text{univ}}_S) \), and let \( J = \ker(Q_S \hat{\otimes}_O \Lambda \to T^T(U(\infty), O)_m) \). To complete the proof of the proposition, we must show that \( I \subset J \). In the proof of [Ger] Corollary 3.1.4] is constructed a Zariski dense family of maximal ideals \( p \) of \( T^T(U(\infty), O)_m[1/l] \) with the following property:

- For each ideal \( p \) there is a finite extension \( E_p \) of \( T^T(U(\infty), O)_m[1/l] / p \) with ring of integers \( O_p \) and a lifting \( r_p : G_{L^+, S} \to G_n(O_p^0) \), where \( O_p^0 \subset O_p \) is the subring of elements whose image in the residue field lies in \( k \), such that \( r_p \) is of type \( S \) and the following diagram commutes:

\[
\begin{CD}
Q_S \hat{\otimes}_O \Lambda @>>> T^T(U(\infty), O)_m \\
\downarrow @VVV \downarrow @VVV \\
R^{\text{univ}}_S @>>> O_p^0.
\end{CD}
\]

In particular, identifying \( p \) with a maximal ideal of \( Q_S \hat{\otimes}_O \Lambda[1/l] \), we have \( I[1/l] \subset p \). Since \( T^T(U(\infty), O)_m[1/l] \) is reduced, it follows that \( I[1/l] \subset J[1/l] \). Since \( T^T(U(\infty), O)_m \) is \( \mathcal{O} \)-flat, it follows that \( I \subset J \).

Let us write \( \Delta \) for the Galois group of the maximal abelian extension of \( L \) of exponent \( l \), unramified outside \( l \). If \( t \geq 1 \) is an integer we write \( \Delta_t \) for its quotient, the Galois group of the maximal abelian extension of exponent \( l \) unramified outside \( l \) and of conductor \( t \) at each place of \( L \) above \( l \).
Proposition 4.11. There is a commutative diagram of \( \Lambda \)-algebras

\[
\begin{array}{ccc}
R_{S_{\chi}}^\text{univ} & \xrightarrow{\alpha} & P_{S_{\chi}} \\
\downarrow & & \downarrow \\
R_{S_{\chi}}^\text{univ} \hat{\otimes} \mathcal{O}[\Delta/(c+1)] & \xleftarrow{\chi} & P_{S_{\chi}} \hat{\otimes} \mathcal{O}[\Delta/(c+1)] \\
\end{array}
\]

\[
\begin{array}{ccc}
& & T_{\chi}^T(U((c)), \mathcal{O})_m \\
\downarrow & & \downarrow \\
R_{S_{\chi}}^\text{univ} \hat{\otimes} \mathcal{O}[\Delta/(c+1)] & \xleftarrow{\chi} & P_{S_{\chi}} \hat{\otimes} \mathcal{O}[\Delta/(c+1)] \\
\end{array}
\]

which extends the diagram of Lemma 3.31.

Proof. We construct maps \( T_{\chi}^T(U((c)), \mathcal{O}) \to T_{\chi}^T(U((c)), \mathcal{O}) \otimes \mathcal{O}[[\Delta]/(c+1)] \) for each \( c \geq t \). It will then be clear from the construction that these maps fit into an inverse system, and localizing and passing to the limit gives the diagram of the proposition.

Fix an isomorphism \( \iota : \mathbb{T}_{\mathrm{q}} \cong \mathbb{C} \), and let \( P_v \) denote the set of automorphic representations \( \pi \) of \( G(\mathbb{A}_{\mathbb{K}}) \) such that \( \pi^{\mathbb{U}((c))} \neq 0 \), and \( \pi \) is \( \iota \)-ordinary. If \( \psi : \Delta/(c+1) \to \mathbb{T}_{\mathrm{q}}^\times \chi \) is a character and \( \chi \in P_v \), then there is an automorphic representation \( \pi \otimes \psi \in P_v \) of \( G(\mathbb{A}_{\mathbb{K}}) \) which is uniquely characterized by the requirement that \( r_{\chi}(\iota^{-1}(\pi \otimes \psi)^\iota) \cong r_{\chi}(\iota^{-1}(\pi \otimes \psi)^\iota) \), where \( \iota^{-1}(\pi \otimes \psi)^\iota \) is as defined in [CHT08 Proposition 3.3.2]. The Hecke algebra \( T_{\chi}^T(U((c)), \mathcal{O}) \) can be identified with the \( \mathcal{O} \)-subalgebra of \( \prod_{\pi \in P_v} \mathbb{T}_{\mathrm{q}} \) generated by the images of the Hecke operators \( T_u^{\text{univ}} \), \( (T_w^{\text{univ}})^{-1} \) and \( (u) \) as above, where the image of each of these operators in the \( \mathbb{T}_{\mathrm{q}} \)-summand corresponding to \( \pi \in P_v \) is its eigenvalue on \( (\iota^{-1}(\pi \otimes \psi)^\iota) \). Let \( f_{\psi} : \prod_{\pi \in P_v} \mathbb{T}_{\mathrm{q}} \to \prod_{\pi \in P_v} \mathbb{T}_{\mathrm{q}} \chi \) denote the automorphism induced by the permutation \( \pi \to \pi \otimes \psi \) of the finite set \( P_v \). An easy calculation shows that if \( \psi \) takes values in \( \mathcal{O}^\chi \) then \( f_{\psi} \) takes the generators of \( T_{\chi}^T(U((c)), \mathcal{O}) \) to \( \mathcal{O}^\times \)-multiples of themselves, and so induces an automorphism of this Hecke algebra. In any case, the product of these homomorphisms over all characters \( \psi : \Delta/(c+1) \to \mathbb{T}_{\mathrm{q}}^\times \chi \) gives a homomorphism

\[
T_{\chi}^T(U((c)), \mathcal{O}) \to \prod_{\psi : \Delta/(c+1) \to \mathbb{T}_{\mathrm{q}}^\times} T_{\chi}^T(U((c)), \mathcal{O}) \otimes \mathbb{T}_{\mathrm{q}} \chi \otimes \mathcal{O}[\Delta/(c+1)]
\]

which actually takes values in the subring \( T_{\chi}^T(U((c)), \mathcal{O}) \otimes \mathcal{O}[\Delta/(c+1)] \). Under this homomorphism, the Hecke operator \( T_u^{\text{univ}} \) is mapped to \( T_u^{\text{univ}} \otimes \text{Frob}_{\mathrm{q}} \). This concludes the proof.

Corollary 4.12. Let \( p \subset R_{S_{\chi}}^\text{univ} \) be a dimension one prime of characteristic \( l \), and let \( J_{S_{\chi}} = \ker(P_{S_{\chi}} \to T_{\chi}^T(U((c)), \mathcal{O})_m) \). Let \( A \) denote the normalization of \( R_{S_{\chi}}^\text{univ}/p \) in its fraction field \( E \), and let \( \psi : \Delta \to 1 + m_A \subset A^\times \) be a continuous character such that \( \psi^p = 1 \). Let \( p_{\psi} \) denote the ideal obtained by twisting \( p \) by \( \psi \) (see Lemma 3.31). Suppose that \( J_{S_{\chi}} \subset p_{\psi} \). Then \( J_{S_{\chi}} \subset p_{\psi} \).

Proof. The ideal \( p_{\psi} \) is the kernel of the composite homomorphism

\[
R_{S_{\chi}}^\text{univ} \to R_{S_{\chi}}^\text{univ} \hat{\otimes} \mathcal{O}[\Delta/(c+1)] \to A,
\]

the homomorphism \( \mathcal{O}[\Delta/(c+1)] \to A \) being induced by the character \( \psi \). The previous proposition now implies that \( J_{S_{\chi}} \) is mapped to zero by this homomorphism.

4.4 Auxiliary levels

We continue with the assumptions of the previous section. Let \( q, N \) be positive integers and fix a choice of Taylor-Wiles data

\[
(Q_N, \eta_N, \{\eta_v\}_{v \in Q_N})
\]

of order \( q \) and level \( N \), where \( \eta_v \) is an eigenvalue of \( \eta_m(\text{Frob}_v) \) of multiplicity \( n_v \). We suppose that for each \( v \in Q_N \), \( n_v \) is coprime to \( l \). In this case we have defined an auxiliary deformation problem (cf. the discussion preceding Lemma 3.17)

\[
S_{\chi,N} = \left( L/L^+, S_N, \delta_N, \Lambda, \eta_m, \epsilon^{-n_\eta} \delta^n L/L^+, \right)
\]
Let $\Delta_N$ denote the maximal $l$-power order quotient of $\prod_{v \in Q_N} k(\overline{v})$. For $v \in Q_N$ we let $\mathfrak{p}^{\overline{v}, v}_N \subset \text{GL}_n(O_{L,v})$ denote the standard parahoric subgroup corresponding to the partition $n = (n - n_v) + n_v$, and $\mathfrak{p}^{\overline{v}, 0, v}_{N,1}$ denote the kernel of the homomorphism

$$\mathfrak{p}^{\overline{v}, v}_N \to \text{GL}_{n_v}(O_{L,v}) \to k(\overline{v}) \to k(\overline{v})(l).$$

Thus $\prod_{v \in Q_N} \mathfrak{p}^{\overline{v}, v}_N / \mathfrak{p}^{\overline{v}, 0, v}_{N,1} \cong \Delta_N$. Finally we set $U_0(\mathcal{Q}_N) = \prod_{v} U_0(\mathcal{Q}_N)_v$, where $U_0(\mathcal{Q}_N)_v = U_v$ if $v \notin Q_N$ and $U_0(\mathcal{Q}_N)_v = \mathfrak{p}^{\overline{v}, v}_N$ if $v \in Q_N$. We set $U_1(\mathcal{Q}_N) = \prod_{v} U_1(\mathcal{Q}_N)_v$, where $U_1(\mathcal{Q}_N)_v = U_v$ if $v \notin Q_N$ and $U_1(\mathcal{Q}_N)_v = \mathfrak{p}^{\overline{v}, 0, v}_{N,1}$ if $v \in Q_N$.

We have a diagram of $\Lambda$-algebras

$$T_{\chi}^{T \cup \mathcal{Q}_N}(U_1(\mathcal{Q}_N)(\mathcal{O}), \mathcal{O}) \to T_{\chi}^{T \cup \mathcal{Q}_N}(U_0(\mathcal{Q}_N)(\mathcal{O}), \mathcal{O}) \to T_{\chi}^{T \cup \mathcal{Q}_N}(U(\mathcal{O}), \mathcal{O})$$

The first two maps are surjective, the third is injective. In an abuse of notation, we write $\mathfrak{m}$ for the pullback of the maximal ideal $\mathfrak{m}$ to each of these algebras. After localizing at $\mathfrak{m}$ the third map becomes an isomorphism, since each localized Hecke algebra can be viewed as the image of the universal pseudo-deformation ring $\mathcal{P}_{\mathcal{S}_N}$.

Fix for each $v \in Q_N$ a choice of uniformizer $\overline{u}_v$ of $O_{L,v}$. In [Tho12] §5 we have constructed operators $\text{pr} = \prod_{v \in Q_N} \text{pr}_{\overline{u}_v}$ which act on the spaces $S_{\chi}(U(\mathcal{O}), \mathcal{O})_m$, $S_{\chi}(U_0(\mathcal{Q}_N)(\mathcal{O}), \mathcal{O})_m$, and $S_{\chi}(U_1(\mathcal{Q}_N)(\mathcal{O}), \mathcal{O})_m$ compatibly with all inclusions and in a manner commuting with the action of the Hecke operators at primes not in $Q_N$. See in particular the proof of [Tho12] Theorem 8.6. We set

$$H_{\chi} = S_{\chi}(U(\mathcal{O}), \mathcal{O})_m = (S_{\chi}(U(\mathcal{O}), K/\mathcal{O})_m)^{\vee},$$

$$H_{\chi,N,0} = (\text{pr}_{\chi} S_{\chi}(U_0(\mathcal{Q}_N)(\mathcal{O}), \mathcal{O})_m)^{\vee},$$

$$H_{\chi,N,1} = (\text{pr}_{\chi} S_{\chi}(U_1(\mathcal{Q}_N)(\mathcal{O}), \mathcal{O})_m)^{\vee}.$$  

For $v \in Q_N$, $\alpha \in k(\overline{v})^{\times}$, we define a Hecke operator $V_{\alpha}$ on $S_{\chi}(U_1(\mathcal{Q}_N)(\mathcal{O}), \mathcal{O})^{\vee}$ as follows. Let $\widetilde{\alpha} \in O_{L,v}$ be any lift of $\alpha$. Then we define

$$V_{\alpha} = \gamma^{-1} \left( \begin{array}{cc} 1_{n - n_v} & 0 \\ 0 & \text{diag}(\widetilde{\alpha}, 1, \ldots, 1) \end{array} \right) \mathfrak{p}^{\overline{v}, 0, v}_{N,1}.$$  

This definition is independent of the choice of $\widetilde{\alpha}$. This operator preserves the spaces $H_{\chi,N,1}$. For $\alpha = (\alpha_i) \in \Delta_N$, we set $V_\alpha = \prod_i V_{\alpha_i}$. The assignment $\alpha \mapsto V_\alpha$ defines an action of the group $\Delta_N$ on the spaces $H_{\chi,N,1}$.

**Theorem 4.13.** (i) The operator $\text{pr}$ induces an isomorphism $H_{\chi,N,0} \cong H_{\chi}$ of $T_{\chi}^{T \cup \mathcal{Q}_N}(U_0(\mathcal{Q}_N)(\mathcal{O}), \mathcal{O})_m$-modules.

(ii) $H_{\chi,N,1}$ is free over $\Lambda[\Delta_N]$ and we have a canonical isomorphism

$$(H_{\chi,N,1})_\Delta \cong H_{\chi,N,0}$$

induced by restriction.

(iii) Let $T_{\chi} = T_{\chi}(U(\mathcal{O}), \mathcal{O})_m$ and let $T_{\chi,N,i}$ denote the image of the map

$$T_{\chi}^{T \cup \mathcal{Q}_N}(U_1(\mathcal{Q}_N)(\mathcal{O}), \mathcal{O})_m \to \text{End}_\Lambda(H_{\chi,N,i}).$$

Then the natural map $T_{\chi} \to T_{\chi,N,0}$ is an isomorphism and for each $\alpha \in \Delta_N$, $V_\alpha \in \text{End}_\Lambda(H_{\chi,N,1})$ actually lies in $T_{\chi,N,1}$.

(iv) The natural homomorphism $\Lambda[\Delta_N] \to R^{\text{univ}}_{S_{\chi,N}}$ factors through $P_{S_{\chi,N}} \subset R^{\text{univ}}_{S_{\chi,N}}$.  

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(v) The pseudodeformation $D_m : G_{L,S} \to T_{X,N,1}$ constructed above induces a surjective map $P_{S_{x,N}} \to T_{X,N,1}$.

**Proof.** The first two points can be deduced from the finite level case just as in the proof of [Tho12, Theorem 8.6](https://example.com). The fifth point is proved just as in the proof of Proposition 4.10. The fourth point follows since for $v \in Q_N$, $\sigma \in I_{L_v}$, the universal trace in $R_{S_{X,N}}^{uni}$ has the form $(n - n_v) + n_v \phi(\sigma)$, where $\phi : I_{L_v} \to (R_{S_{X,N}}^{uni})^\times$ is the character through which, by construction, inertia at $\bar{v}$ acts on the universal deformation of type $S_{X,N}$. Since $l$ does not divide $n_v$ by hypothesis, we have $\phi(\sigma) \in P_{S_{x,N}}$. This also implies the third part. □

4.5 Soluble base change

We put ourselves in the setting of §4.5. In this section we suppose that $M/L$ is a soluble CM extension, linearly disjoint over $L$ from the extension of $L(\zeta)$ cut out by $\tau_m|_{G_{L(\zeta)}}$, in which every prime above $S_t \cup S_n \cup R$ splits. (Thus $M/L$ is a good extension, in the language of the proof of Theorem 6.1 below.) We write $T_M$ (resp. $\hat{T}_M$) for the set of primes above of $M^+$ (resp. $M$) above $T$ (resp. $\hat{T}$). We write $S_{t,M}$, $R_M$, $S(B)_M$, $S_{a,M}$ for the sets of primes of $M^+$ above the sets $S_t$, $R$, $S(B)$, and $S_a$, respectively.

Let $\Lambda_M = \hat{\otimes}_{v \in S_{t,M}} \mathcal{O}[I_{M_{ab}}(l)^n]$ denote the Iwasawa algebra of $M$. There is a natural homomorphism $\Lambda_M \to \Lambda$, induced by the norm homomorphism $(\mathcal{O}_{M^+} \otimes \mathbb{Z}_l)^\times \to (\mathcal{O}_{L^+} \otimes \mathbb{Z}_l)^\times$. We will frequently use the following simple lemma.

**Lemma 4.14.** Suppose that $A \in C_O$, and for each $v \in S_t$, suppose given continuous characters $\varphi_1^v, \ldots, \varphi_n^v : I_{L_v} \to A^\times$. Then the homomorphism $\Lambda_M \to A$ induced by the restrictions $\varphi_i^v|_{G_{M_v}}$ for $u \in S_{t,M}$ dividing $v \in S_t$ factors through the homomorphism $\Lambda_M \to A$ above.

**Proof.** We reduce immediately to the universal case $A = \Lambda$. In this case the lemma follows from the commutativity of the diagram

$$
\begin{array}{ccc}
M^\times & \xrightarrow{\text{Art}_{M_{ab}}} & G_{M_{ab}}^\times \\
\downarrow{\text{N}_{M_v/L_v}} & & \downarrow{\text{N}_{L_v/L_v}} \\
L_v^\times & \xrightarrow{\text{Art}_{L_v}} & G_{L_v}^\times 
\end{array}
$$

The characters $\chi_{v,i} : k(\overline{v})^\times \to A^\times$ for $v \in R$ induce characters of the groups $k(\overline{v})^\times$ for $v \in R_M$. By abuse of notation, we denote this collection of characters for $v \in R_M$ by $\chi$. We can then define a new deformation problem

$$S_{X,M} = \left( M/M^+, T_M, \hat{T}_M, \Lambda_M, \tau_m|_{G_{M^+}}, \varphi_{1,v}^{\Lambda^v} \right)_{v \in S_{t,M}} \cup \left( R_{v}^{\Lambda^v} \right)_{v \in R_M}$$

Thus the universal deformation ring $R_{S_{X,M}}^{uni}$ and its $\Lambda_M$-subalgebra $P_{S_{x,M}}$ are defined. We write $U_M = \prod U_{v,M}$ for the (sufficiently small) open compact subgroup of $G(A_{M^+}^\infty)$ defined as follows:

- For $v$ inert in $M$, $U_v \subset G(M_v^+)$ is a hyperspecial maximal compact subgroup.
- For $v \notin T_M$ split in $M$, $U_v = G(O_{M_v^+})$.
- For $v \in S(B)_M$, $U_v$ is a maximal compact subgroup.
- For $v \in S_{t,M}$, $U_v = G(O_{M_v^+})$.
- For $v \in R_M$, $U_v = \tau_v^{-1} \text{Iw}(\overline{v})$.
- For $v \in S_{a,M}$, $U_v = \tau_v^{-1} \ker(\text{GL}_n(O_{M_v}) \to \text{GL}_n(k(\overline{v})))$.

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Here for $v \in T_M - S(B)_M$, $\bar{v}_G$ is an isomorphism $G(M_v^+) \cong GL_n(M_v)$, defined in the same way as the isomorphism $v_G$ for $v \in T - S(B)$. The Hecke algebra $T_X^M(U_M/(\infty), O)$ is defined in the same way as above. By [AC89, Ch. 3, Theorem 4.2] and [CLOT03 Proposition 3.3.2], there exists a homomorphism $T_X^M(U_M/(\infty), O) \to \overline{\mathbb{F}}_l$, whose kernel we denote by $m_M$, such that $\pi_{\mathfrak{m}_M}|G_M \cong \pi_{\mathfrak{m}}|G_M$. The maximal ideal $\mathfrak{m}_M$ is residually Schur and we can make our choices so that $\pi_{\mathfrak{m}_M} = \pi_{\mathfrak{m}}|G_{M^+}$.

We now have a diagram of $\Lambda_M$-algebras

$$
\begin{array}{ccc}
R_{\chi,M}^{univ} & \longrightarrow & P_{\chi,M} \\
\downarrow & & \downarrow \\
\overline{T}_X^M(U_M/(\infty), O)_{\mathfrak{m}_M} & \longrightarrow & \overline{T}_X^M(U_M/(\infty), O). \\
\end{array}
$$

**Proposition 4.15.** Restriction of deformations $r \mapsto r|_{G_{M^+}}$ induces a map $R_{\chi,M}^{univ} \to R_{\chi}^{univ}$. This is a finite homomorphism of $\Lambda_M$-algebras, where $R_{\chi}^{univ}$ is given a $\Lambda_M$-algebra structure by the homomorphism $\Lambda_M \to \Lambda$.

**Proof.** The existence of the map $R_{\chi,M}^{univ} \to R_{\chi}^{univ}$ is a local problem. Since we chose $M$ to be split above the primes in $S_T \cup S_u \cup R$, it remains to show that if $w \in S(B)_M$ is a place dividing $v \in S(B)$, then the restriction of the universal lifting induces a natural map $R_{\chi,w}^{univ} \to R_{\chi}^{univ}$. However, this is immediately apparent from the definitions. We now obtain a commutative diagram of $\Lambda_M$-algebras

$$
\begin{array}{ccc}
P_{\chi,M} & \longrightarrow & P_{\chi} \\
\downarrow & & \downarrow \\
R_{\chi,M}^{univ} & \longrightarrow & R_{\chi}^{univ}. \\
\end{array}
$$

To prove the proposition, it suffices to prove that the ring $R_{\chi,M}^{univ}/\mathfrak{m}_{R_{\chi,M}^{univ}}$ is an Artinian $k$-algebra. This ring classifies deformations of $\pi_{\mathfrak{m}}$ containing a lifting whose restriction to $G_{M^+}$ equals $\pi_{\mathfrak{m}}|G_{M^+}$. Let $M_0$ denote the extension of $M^+$ cut out by $\pi_{\mathfrak{m}}|G_{M^+}$. Then any such lifting is trivial on the finite index subgroup $G_{M_0}$ of $G_{M^+}$. Let $r_{\chi,M}^{univ}$ denote a representative of the universal deformation. If $\mathfrak{p} \subset R_{\chi,M}^{univ}/\mathfrak{m}_{R_{\chi,M}^{univ}}$ is a prime ideal, then the coefficients of the characteristic polynomial of $r_{\chi,M}^{univ}(\sigma) \mod \mathfrak{p}$ for $\sigma \in G_L$ are amongst the sums of roots of unity of bounded order, so are finite in number. Arguing as in the proof of [BLGGT08 Lemma 1.2.2], we see that the subring of $R_{\chi,M}^{univ}/\mathfrak{p}$ topologically generated by these elements is a finite $k$-algebra, and hence $R_{\chi,M}^{univ}/\mathfrak{p}$ is itself a finite $k$-algebra (being finite over the subring topologically generated by the coefficients of these characteristic polynomials, by Proposition 3.25). It follows that $R_{\chi,M}^{univ}/\mathfrak{m}_{R_{\chi,M}^{univ}}$ is a $k$-algebra of dimension 0, hence an Artinian $k$-algebra.

**Proposition 4.16.** There is a commutative diagram of $\Lambda_M$-algebras

$$
\begin{array}{ccc}
R_{\chi,M}^{univ} & \longrightarrow & P_{\chi} \\
\downarrow & & \downarrow \\
R_{\chi,M}^{univ} & \longrightarrow & \overline{T}_X^M(U_M/(\infty), O)_{\mathfrak{m}_M}. \\
\end{array}
$$

In particular, if we write $J_{\chi,M} = \ker(P_{\chi} \to \overline{T}_X^M(U_M/(\infty), O)_{\mathfrak{m}_M})$ and similarly for $J_{\chi,M}$, then we have $J_{\chi,M}P_{\chi} \subset J_{\chi}$. 

**Proof.** It remains to construct a map $\overline{T}_X^M(U_M/(\infty), O)_{\mathfrak{m}_M} \to \overline{T}_X^M(U_M/(\infty), O)$ and show that the right-hand square commutes. For each integer $c > 0$, let $\mathfrak{m}_{c,M}$ and $\mathfrak{m}_c$ denote the images of the maximal ideals $\mathfrak{m}_M$ and $\mathfrak{m}$ in the Hecke algebras $\overline{T}_X^M(U_M/(c), O)$ and $\overline{T}_X^M(U/(c), O)$. It suffices to construct maps $\overline{T}_X^M(U_M/(c), O) \to \overline{T}_X^M(U/(c), O)$ and show that the resulting square

$$
\begin{array}{ccc}
P_{\chi} & \longrightarrow & \overline{T}_X^M(U/(c), O)_{\mathfrak{m}_c} \\
\downarrow & & \downarrow \\
P_{\chi,M} & \longrightarrow & \overline{T}_X^M(U_M/(c), O)_{\mathfrak{m}_{c,M}}. \\
\end{array}
$$

28
commutes. Fix an isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, and write $P_c$ for the set of automorphic representations $\pi$ of $G(\mathbb{A}_{L+})$ which contribute to the space $S_\chi(U(\mathbb{C}, c), \mathcal{O})_{m \pi}$. Similarly, write $P_{c,M}$ for the set of automorphic representations of $G(\mathbb{A}_{M+})$ which contribute to the space $S_\chi(U_M(\mathbb{C}, c), \mathcal{O})_{m \pi,M}$. The algebra $T_\chi^T(U(\mathbb{C}, c), \mathcal{O})_{m \pi}$ may be identified with the $\mathcal{O}$-subalgebra of $\prod_{\pi \in P_c} \overline{\mathbb{Q}}_l$ generated by the images of the Hecke operators described in §4.1 and also with the image of the natural ring homomorphism $P_{S_\chi} \to \prod_{\pi \in P_c} \overline{\mathbb{Q}}_l$. For each $\pi \in P_c$, we may choose an automorphic representation $\pi_M \in P_{c,M}$ such that $\tau_l(l^{-1}\pi)|G_M \cong \tau_l(l^{-1}\pi_M)$. This choice gives rise to a commutative diagram:

$$
\begin{array}{ccc}
P_{S_\chi} & \overset{\pi}{\longrightarrow} & \prod_{\pi \in P_c} \overline{\mathbb{Q}}_l \\
\downarrow \ & & \downarrow \\
P_{S_\chi, M} & \overset{\pi_M}{\longrightarrow} & \prod_{\pi \in P_c, M} \overline{\mathbb{Q}}_l
\end{array}
$$

and identifying the images of the horizontal arrows with the respective Hecke algebras gives the desired commutative square.

### 4.6 A patching argument

We put ourselves in the setting of §4.3. We leave the characters $\chi_v$ for $v \in R$ unspecified, but suppose that they are all trivial mod $\lambda$. Note that the rings $R^{\text{univ}}_{S_\chi}/\Lambda$ for varying $\chi$ are canonically identified. Similar remarks apply to the rings $P_S$, $P_{S_n,T}$ etc. and the spaces of automorphic forms $S_\chi(U(\mathbb{C}), \mathcal{O})_{m \pi}$. In particular, the sets of prime ideals containing $\lambda$ in each of these rings are in canonical bijection. In the following we will abuse notation and view such a prime ideal as belonging to any one of these rings, depending on the context. The level $U$ will be fixed throughout this section, so we write $H_\chi = S_\chi(U(\mathbb{C}), \mathcal{O})_m$ and $T_\chi = T_\chi^T(U(\mathbb{C}), \mathcal{O})_{m \pi}$ for this space of automorphic forms and Hecke algebra, respectively, as in §4.4. We will also adopt the notations $H_{\chi,N,1}$ and $T_{\chi,N,1}$ of that section for the corresponding objects at auxiliary levels, once we make a choice of Taylor-Wiles data of level $N$.

Let $J_{S_\chi} = \ker(P_{S_\chi} \to T_\chi)$. We suppose that $\mathfrak{p} \supset J_{S_\chi} R^{\text{univ}}_{S_\chi}$ is a prime ideal of dimension one and characteristic $l$. Write $A$ for the normalization of $R^{\text{univ}}_{S_\chi}/\mathfrak{p}$ in its fraction field $E$. We let $\mathfrak{q} = P_{S_\chi} \cap \mathfrak{p}$. Suppose that the map $\Lambda \to A$ is finite. Arguing as in §3.7 we can choose a finite faithfully flat extension $\Lambda \to \widetilde{\Lambda}$ inducing a bijection on minimal primes, together with a surjective map $\Lambda \to A$ with kernel $\widetilde{\mathfrak{p}}$ and making the diagram

$$
\begin{array}{ccc}
\Lambda & \longrightarrow & \widetilde{\Lambda} \\
\downarrow \ & & \downarrow \\
& A
\end{array}
$$

commute, and extensions $\widetilde{\mathfrak{p}}$ (resp. $\widetilde{\mathfrak{q}}$) of $\mathfrak{p}$ (resp. $\mathfrak{q}$) to the rings $R^{\text{univ}}_{S_\chi} \otimes_\Lambda \widetilde{\Lambda}$ (resp. $P_{S_\chi} \otimes_\Lambda \widetilde{\Lambda}$).

We set $R^{\text{univ}}_{S_\chi} = R^{\text{univ}} \otimes_\Lambda \widetilde{\Lambda}$, $P_{S_\chi} = P_{S_\chi} \otimes_\Lambda \widetilde{\Lambda}$, and $\widetilde{T}_\chi = T_\chi \otimes_\Lambda \widetilde{\Lambda}$. We set $T = \Lambda[[X_1, \ldots, X_{n+1}]]$. For each $\chi$ we fix a choice of lifting $r^{\text{univ}}_{\chi} : G_{L+} \to G_n(R^{\text{univ}}_{S_\chi})$ representing the universal deformation $[r^{\text{univ}}_{S_\chi}]$. We suppose that the different choices are identified modulo $\lambda$. Having made this choice we can write down an isomorphism $R^{\text{univ}}_{S_\chi} \cong R^{\text{univ}}_{S_\chi} \otimes_\Lambda T$. We define $P^{\otimes}_{S_\chi} = P_{S_\chi} \otimes_\Lambda T$ so that there is a commutative diagram

$$
\begin{array}{ccc}
R^{\text{univ}}_{S_\chi} & \longrightarrow & R^{\text{univ}}_{S_\chi} \otimes_\Lambda T \\
\downarrow \ & & \downarrow \\
P_{S_\chi} & \longrightarrow & P_{S_\chi} \otimes_\Lambda T
\end{array}
$$

For varying $\chi$ these diagrams are identified modulo $\lambda$, by the choice of universal lifting. For any choice of
Taylor-Wiles data of level $q$ and level $N$, we obtain a diagram

$$
\begin{array}{c}
P_{S_{x,N}}^{\square_T} \rightarrow P_{S_{x}} \rightarrow A \\
\downarrow \quad \quad \quad \downarrow \\
R^{\text{loc}}_{S_{x,T}} \rightarrow R^{\square_T}_{S_{x,N}} \rightarrow R^{\univ}_{S_{x}} \rightarrow A
\end{array}
$$

We write $p_N$, $q_N$ for the kernels of the respective maps $R^{\square_T}_{S_{x,N}} \rightarrow A$ and $P_{S_{x,N}}^{\square_T} \rightarrow A$. We write $P^{\text{loc}}$ for the kernel of the map $R^{\text{loc}}_{S_{x,T}} \rightarrow A$. We define

$$
\tilde{R}^{\square_T}_{S_{x,N}} = R^{\square_T}_{S_{x,N}} \otimes_{\Lambda} \tilde{\Lambda}, \quad \tilde{P}^{\square_T}_{S_{x,N}} = P^{\square_T}_{S_{x,N}} \otimes_{\Lambda} \tilde{\Lambda}, \quad \tilde{R}^{\text{loc}}_{S_{x,T}} = \tilde{R}^{\text{loc}}_{S_{x,T}} \otimes_{\Lambda} \tilde{\Lambda},
$$

and let $\tilde{p}_N$, $\tilde{q}_N$, and $\tilde{P}^{\text{loc}}$ denote the natural extensions of the ideals $p_N$, $q_N$, and $P^{\text{loc}}$ to prime ideals of these rings. We also define

$$
\tilde{P}_{S_x} = P_{S_x} \otimes_{\Lambda} \tilde{\Lambda}, \quad \tilde{R}^{\univ}_{S_x} = R^{\univ}_{S_x} \otimes_{\Lambda} \tilde{\Lambda}, \quad \tilde{T} = \mathcal{T} \otimes_{\Lambda} \tilde{\Lambda}.
$$

**Theorem 4.17.** Suppose that $p$ satisfies the following hypotheses:

(i) There exists an integer $q \geq [L^+: \mathbb{Q}]n(n-1)/2$ and for each positive integer $N$, a choice of Taylor-Wiles data of order $q$ and level $N$ as above such that there is an isomorphism of $A$-modules

$$
\tilde{p}_N/(\tilde{P}^{\text{loc}} + \tilde{p}_N^2) \cong A(q-[L^+:\mathbb{Q}]n(n-1)/2) \oplus T(N),
$$

where $T(N)$ is finite of cardinality bounded independently of $N$.

(ii) $\pi_m|G_{P,S} = \pi_1 \oplus \pi_2$ is a direct sum of two absolutely irreducible representations.

(iii) For each $v|l$, the pushforwards of the universal characters $\psi_1^v, \ldots, \psi_n^v$ to $A$ are pairwise distinct, $q_v \equiv 1 \mod l$ and $[L_v: \mathbb{Q}]/[L: \mathbb{Q}] > n(n+1)/2 + 1$.

(iv) For each $v \in R$, $r_P|G_{L_v}$ is the trivial representation, and if $t^N | q_v - 1$ then $t^N > n$. For each $v \in S(B)$, $r_P|G_{L_v}$ is unramified and $r_P(Frob_v)$ is a scalar matrix.

(v) $\text{Frac} P_{S_x}/q = \text{Frac} R_{S_x}/p$.

Then the map $\tilde{P}_{S_{x,\mathfrak{q}}} \rightarrow \tilde{T}_{1,\mathfrak{q}}$ has nilpotent kernel.

**Corollary 4.18.** With hypotheses as above, let $Q \subset p$ be a minimal prime of $R^{\univ}_{S_{x,\mathfrak{p}}}$ Then $J_{S_{x}}R^{\univ}_{S_{x,\mathfrak{p}}} \subset Q$.

**Proof.** The theorem shows that the ideals $J_{S_{x}}\tilde{P}_{S_{x,\mathfrak{q}}}$ and hence $J_{S_{x}}\tilde{R}^{\univ}_{S_{x,\mathfrak{p}}}$ are nilpotent. The corollary now follows since the map $R^{\univ}_{S_{x,\mathfrak{p}}} \rightarrow (\tilde{R}^{\univ}_{S_{x,\mathfrak{p}}}/p)$ is faithfully flat. $\square$

The remainder of this section is devoted to the proof of the above theorem. We fix an integer $q$ and for each integer $N \geq 1$ a choice of Taylor-Wiles data as in the statement of the theorem. We define $\tilde{H}_{\chi} = H_{\chi} \otimes_{\Lambda} \tilde{\Lambda}$, and introduce auxiliary Hecke modules

$$
\tilde{H}^{\square}_{\chi} = \tilde{H}_{\chi} \otimes_{\tilde{P}_{S_x}} \tilde{P}^{\square_T}_{S_x}, \quad \tilde{H}^{\square}_{\chi,N} = \tilde{H}^{\square}_{\chi,N,1} \otimes_{\tilde{P}_{S_x}} \tilde{P}^{\square_T}_{S_x}
$$

Then $\tilde{H}^{\square}_{\chi,N}$ is a free $\tilde{T}[\Delta_N]$-module, with $\Delta_N$-covariants isomorphic to $\tilde{H}^{\square}_{\chi}$.

We now set $S_{\infty} = \tilde{T}[S_1, \ldots, S_q]$, and write $a = \ker(S_{\infty} \rightarrow \Lambda)$, $P_{\infty} = \ker(S_{\infty} \rightarrow \Lambda \rightarrow A)$. We choose for every $N$ a surjection $S_{\infty} \rightarrow \tilde{T}[\Delta_N]$, and write $\epsilon_N = \ker(S_{\infty} \rightarrow \tilde{T}[\Delta_N])$. With these choices $\tilde{P}^{\square_T}_{S_{x,N}}$, $\tilde{R}^{\square_T}_{S_{x,N}}$ become $S_{\infty}$-algebras for every $N$, and $\tilde{H}^{\square}_{\chi,N}$ is a free $S_{\infty}/\epsilon_N$-module. We write $\mathfrak{b}_N \subset S_{\infty}$ for the ideal $(m^N_{\chi}, (S_1+1)^{N_1} - 1, \ldots, (S_q+1)^{N_q} - 1, X_1^{N_1}, \ldots, X_n^{N_n})$. Thus $\epsilon_N \subset \mathfrak{b}_N$ and $\tilde{H}^{\square}_{\chi,N}/\mathfrak{b}_N$ is a free $S_{\infty}/\mathfrak{b}_N$-module.
Let $q' = (q - [L^+ : \mathbb{Q}]n(n-1)/2)$. We define

$$R_{\chi}^\infty = \widetilde{R}_{\mathcal{S},T}^\text{loc} \llbracket y_1, \ldots, y_q \rrbracket.$$ 

We fix for every $\chi$ and $N \geq 1$ a homomorphism of $\widetilde{R}_{\mathcal{S},N}$-algebras

$$R_{\chi}^\infty \to \widetilde{R}_{\mathcal{S},N}^\text{loc}$$

taking $y_1, \ldots, y_{q'}$ onto a basis of the maximal free quotient of the $A$-module $\widetilde{p}_N/(\widetilde{p}_{2N}^\text{loc} + \widetilde{p}_N^2)$. We suppose that these are chosen to be identified upon reduction modulo $\chi$. We write $P^\infty$ for the kernel of the surjective homomorphism $R_{\chi}^\infty \to A$. Thus $P^\infty$ is the pullback of $\widetilde{P}_{\mathcal{S},T}^\text{loc}$ along the natural homomorphism $R_{\chi}^\infty \to \widetilde{R}_{\mathcal{S},T}^\text{loc}$.

We recall (cf. Proposition 3.33) that the group $\mu_2 \times \mu_2$ acts on the rings $\widetilde{P}_{S,N}^\text{univ}, \widetilde{R}_{S,N}^\text{univ}, \widetilde{P}_{\mathcal{S}}$, and $\widetilde{R}_{\mathcal{S}}^\text{univ}$, compatibly with the maps between these objects and identifications modulo $\lambda$.

Let $r_M = M(q + n^2)t_M$, where $s = \dim_k \widetilde{H}_{\chi}/m_{\chi}$. (Note that $s$ is independent of the choice of $\chi$.) For any integer $M \geq 1$ we define a patching datum $(A_{\chi,M}, B_{\chi,M}, \mathcal{M}_{\chi,M})_\chi$ of level $M$ to be for each choice of $\chi$ a commutative diagram of complete Noetherian local $\Lambda$-algebras with residue field $k$:

$$S_{\chi} \longrightarrow A_{\chi,M} \overset{\phi_P}{\longrightarrow} \widetilde{P}_{\chi,M}/(m_{P_{S_{\chi}}}^M + b_M) \overset{\phi_M}{\longrightarrow} R_{\chi}^\infty \longrightarrow B_{\chi,M} \overset{\phi_B}{\longrightarrow} \widetilde{R}_{\mathcal{S},N}^\text{univ}/(m_{P_{S_{\chi}}}^M + b_M)$$

together with an $A_{\chi,M}$-module $\mathcal{M}_{\chi,M}$ killed by $b_M$ and a homomorphism $\psi_{\chi} : \mathcal{M}_{\chi,M} \to \widetilde{H}_{\chi}/b_M$ inducing an isomorphism $\mathcal{M}_{\chi,M}/a \cong \widetilde{H}_{\chi}/b_M$. We fix also the data of identifications between these diagrams and modules for varying $\chi$. We require further that $\mathcal{M}_{\chi,M}$ be finite free as an $S_{\chi}/b_M$-module. We also fix the data of an action of the group $\mu_2 \times \mu_2$ on the rings $A_{\chi,M}$ and $B_{\chi,M}$ such that the arrows in the right-hand square of the above diagram are equivariant for the action of this group. A morphism of patching data $(A_{\chi,M}, B_{\chi,M}, \mathcal{M}_{\chi,M})_\chi, (A'_{\chi,M}, B'_{\chi,M}, \mathcal{M}'_{\chi,M})_\chi$ of level $M$ is the data for each $\chi$ of isomorphisms $A_{\chi,M} \to A'_{\chi,M}$, $B_{\chi,M} \to B'_{\chi,M}$ compatible with the action of $\mu_2 \times \mu_2$ and a compatible isomorphism of modules $\mathcal{M}_{\chi,M} \to \mathcal{M}'_{\chi,M}$, making the diagram

$$S_{\chi} \longrightarrow A_{\chi,M} \longrightarrow A'_{\chi,M} \longrightarrow \widetilde{P}_{\chi,M}/(m_{P_{S_{\chi}}}^M + b_M) \longrightarrow R_{\chi}^\infty \longrightarrow B_{\chi,M} \longrightarrow B'_{\chi,M} \longrightarrow \widetilde{R}_{\mathcal{S},N}^\text{univ}/(m_{P_{S_{\chi}}}^M + b_M)$$

commutative, and such that the identifications mod $\lambda$ are preserved.

For each pair of positive integers $M \leq N$ we can construct a patching datum $D(M,N)$ of level $M$ by taking for each $\chi$ the diagram

$$S_{\chi} \longrightarrow \widetilde{P}_{S_{\chi},N}^\text{loc}/(m_{P_{S_{\chi}}}^M + b_M) \longrightarrow \widetilde{P}_{S_{\chi}}/(m_{P_{S_{\chi}}}^M + b_M) \longrightarrow R_{\chi}^\infty \longrightarrow \widetilde{R}_{S_{\chi},N}^\text{loc}/(m_{P_{S_{\chi}}}^M + b_M) \longrightarrow \widetilde{R}_{S_{\chi}}^\text{univ}/(m_{P_{S_{\chi}}}^M + b_M),$$

with Hecke modules $\widetilde{H}_{\chi,N}/b_M \to \widetilde{H}_{\chi}/b_M$. The action of $\mu_2 \times \mu_2$ on these rings is induced by its action on $\widetilde{P}_{S_{\chi},N}^\text{loc} \cong \widetilde{P}_{S_{\chi}} \otimes \mathcal{A} \bar{T}$ and $\widetilde{R}_{S_{\chi},N}^\text{loc} \cong \widetilde{R}_{S_{\chi}} \otimes \mathcal{A} \bar{T}$, where $\mu_2 \times \mu_2$ is made to act trivially on $\bar{T}$. 31
Lemma 4.19. (i) The ring $\bar{R}_{S_{\chi},N}^{\square}/(m_{P_{S_{\chi},N}}^M + b_M)$ acts on $\bar{H}_{\chi,N}^{\square}/b_M$. In particular, this does indeed define a patching datum of level $M$.

(ii) Fix $M$. Then as $N \geq M$ varies, the patching data $D(M,N)$ fall into finitely many isomorphism classes.

Proof. For the first part it suffices to show that $m_{P_{S_{\chi},N}}^M \bar{R}_{S_{\chi},N}^{\square} \subset b_M \bar{H}_{\chi,N}^{\square}$. Suppose that $x \in m_{P_{S_{\chi},N}}^M$. Then $x$ is nilpotent on the $s$-dimensional $k$-vector space

$$\bar{H}_{\chi,N}^{\square}/(m_{\bar{\chi}} + a)\bar{H}_{\chi,N}^{\square} \cong H_{\chi}/m_{\bar{\chi}},$$

so $x^s$ acts as the zero map. It follows that

$$x^s \bar{H}_{\chi,N}^{\square} \subset (m_{\bar{\chi}}, S_1, \ldots, S_q, X_1^M, \ldots, X_{n^2t}) \bar{H}_{\chi,N}^{\square}$$

and hence

$$x^{(q+n^2t)} \bar{H}_{\chi,N}^{\square} \subset (m_{\bar{\chi}}, S_1^M, \ldots, S_q^M, X_1^M, \ldots, X_{n^2t}^M) \bar{H}_{\chi,N}^{\square} = (m_{\bar{\chi}}, (S_1 + 1)^M - 1, \ldots, (S_q + 1)^M - 1, X_1^M, \ldots, X_{n^2t}^M) \bar{H}_{\chi,N}^{\square}.$$

Then we have

$$x^{M(q+n^2t)} \bar{H}_{\chi,N}^{\square} \subset (m_{\bar{\chi}}, (S_1 + 1)^M - 1, \ldots, (S_q + 1)^M - 1, X_1^M, \ldots, X_{n^2t}^M) \bar{H}_{\chi,N}^{\square},$$

as required.

For the second part, it suffices to show that the orders of the rings $\bar{R}_{S_{\chi},N}^{\square}/(m_{P_{S_{\chi},N}}^M + b_M)$ and $\bar{R}_{S_{\chi},N}^{\square}/(m_{P_{S_{\chi},N}}^M + b_M)$ can be bounded solely in terms of $M$, this being clear for all other objects in the diagrams above. For the quotient of $\bar{R}_{S_{\chi,N}}^{\square}$, this is an immediate consequence of Lemma 3.24. On the other hand, note that $\bar{R}_{S_{\chi,N}}^{\square}/m_{P_{S_{\chi},N}}^M \cong \bar{R}_{S_{\chi,N}}^{\text{univ}}/m_{P_{S_{\chi},N}}^M$, hence for any $N \geq 1 \bar{R}_{S_{\chi,N}}^{\square}$ is generated as a $\bar{P}_{S_{\chi,N}}^{\square}$-module by $\dim_k \bar{R}_{S_{\chi,N}}^{\text{univ}}/m_{P_{S_{\chi},N}}$ elements. The result follows. $\square$

We now patch to obtain objects at ‘infinite level’. For every fixed $M$, the patching data $D(M,N)$ for $N = M, M+1, \ldots$ fall into a finite number of isomorphism classes. Thus we can find an infinite sequence $1 \leq N_1 < N_2 < \ldots$ of integers such that for all fixed $M$, the patching data $D(M,N_j)$ for $j \geq i$ are all equivalent. Choosing for each $M$ an isomorphism of patching data $D(M,N_{M+1}) \cong D(M,N_M)$ we obtain an inverse system and can therefore pass to the limit to obtain for each $\chi$ a diagram of rings

$$\begin{array}{ccc}
S_\infty & \longrightarrow & A_{\chi,\infty} \\
\downarrow & & \downarrow \\
R_\chi & \longrightarrow & B_{\chi,\infty}
\end{array}$$

and a module $M_{\chi,\infty}$ for $A_{\chi,\infty}$ which is free over $S_\infty$ of rank $s$. Moreover for varying $\chi$ these diagrams are identified modulo $\lambda$, and $\mu_2 \times \mu_2$ acts on $A_{\chi,\infty}$ and $B_{\chi,\infty}$ in such a way that the arrows in the right-hand square are equivariant for this action.

Let $q_\infty, p_\infty$ denote respectively the pullback of $\tilde{q}$ and $\tilde{p}$ to $A_{\chi,\infty}$ and $B_{\chi,\infty}$. It follows from Proposition 3.33 that $p_\infty/(q_\infty + p_\infty^2)$ is a finite torsion $A$-module. On the other hand, there is by construction an isomorphism $B_{\chi,\infty}/q_\infty \cong \bar{R}_{S_{\chi}}^{\text{univ}}/\tilde{q}$, compatible with the action of $\mu_2 \times \mu_2$. By Proposition 3.33 this group acts transitively on the set of minimal primes of of $\bar{R}_{S_{\chi}}/\tilde{q}$.

Lemma 4.20. (i) $B_{\chi,\infty}$ is a finite $A_{\chi,\infty}$-algebra, and the map $A_{\chi,\infty} \rightarrow B_{\chi,\infty}$ has nilpotent kernel.
(ii) The map $A_{\chi,\infty, q_\infty} \to B_{\chi,\infty, p_\infty}$ is surjective, with nilpotent kernel.

Proof. For the first part, the finiteness follows from the corresponding fact at finite level, and the completed version of Nakayama’s lemma. To calculate the kernel we use Fitting ideals. Recall that if $R$ is a Noetherian ring and $M$ is an $R$-module that can be generated by $r$ elements, we have $(\operatorname{Ann}_R M)^r \subset \operatorname{Fitt}_R M \subset \operatorname{Ann}_R M$, and for any homomorphism $R \to S$ we have $\operatorname{Fitt}_S M \otimes_R S = \operatorname{Fitt}_R M \cdot S$. For each $M \geq 1$ we have $\operatorname{Ann}_{\tilde{p}^r_{\tilde{S}^r_{\tilde{X},\tilde{N}, M}}} \tilde{R}^r_{\tilde{S}^r_{\tilde{X}, N, M}} = 0$, hence $\operatorname{Fitt}_{\tilde{p}^r_{\tilde{S}^r_{\tilde{X}, N, M}}} \tilde{R}^r_{\tilde{S}^r_{\tilde{X}, N, M}} = 0$, hence

$$\operatorname{Fitt}_{\tilde{p}^r_{\tilde{S}^r_{\tilde{X}, N, M}} / (\tilde{m}^r_{\tilde{p}^r_{\tilde{S}^r_{\tilde{X}, N, M}}} + b_{\tilde{m}})} \tilde{R}^r_{\tilde{S}^r_{\tilde{X}, N, M}} / (\tilde{m}^r_{\tilde{p}^r_{\tilde{S}^r_{\tilde{X}, N, M}}} + b_{\tilde{m}}) = 0,$$

hence

$$\operatorname{Fitt}_{A_{\chi,\infty}} B_{\chi,\infty} = \lim_{\operatorname{Proj}_{\tilde{m}}} \operatorname{Fitt}_{\tilde{S}^r_{\tilde{X}, N, M}} / (\tilde{m}^r_{\tilde{p}^r_{\tilde{S}^r_{\tilde{X}, N, M}}} + b_{\tilde{m}}) \tilde{R}^r_{\tilde{S}^r_{\tilde{X}, N, M}} / (\tilde{m}^r_{\tilde{p}^r_{\tilde{S}^r_{\tilde{X}, N, M}}} + b_{\tilde{m}}) = 0.$$

For the second part, the surjectivity follows from the vanishing of the relative tangent space. To see that the kernel is nilpotent, we note that the map $A_{\chi,\infty, q_\infty} \to B_{\chi,\infty, q_\infty}$ has nilpotent kernel, the product being over primes $q_\infty$ of $B_{\chi,\infty}$ above $q_\infty$. However, for any such prime $q_\infty$, the rings $B_{\chi,\infty, p_\infty}$ and $B_{\chi,\infty, q_\infty}$ are isomorphic $A_{\chi,\infty, q_\infty}$-algebras, by the $\mu_2 \times \mu_2$-action. The result follows.

Lemma 4.21. (i) $\tilde{H}_{\tilde{X}, \tilde{A}}$ is a direct factor of $\tilde{H}_{\tilde{X}, \tilde{B}}$ and is a non-zero free $\tilde{A}_{\tilde{B}}$-module.

(ii) $M_{\chi,\infty, q_\infty}$ is a direct factor of $M_{\chi,\infty, p_\infty}$, and is a non-zero free $S_{\infty, p_\infty}$-module, with $M_{\chi,\infty, q_\infty}/A \simeq \tilde{H}_{\tilde{X}, \tilde{A}}$ compatibly with the action of $\chi$.

(iii) The induced map $R_{\chi, p_\infty}^\infty \to B_{\chi,\infty, p_\infty}$ is surjective.

Proof. For the first part, $\tilde{H}_{\tilde{X}, \tilde{P}} = H_{X} \otimes_{\Lambda} \tilde{A}_{P}$. The action of $\tilde{P}_S$ on $H_{X}$ factors through a quotient $\tilde{T}_X$ which is fine over $\tilde{A}$, hence $\tilde{T}_X \otimes_{\tilde{A}} \tilde{A}_{P}$ has $\tilde{T}_{\tilde{X}, \tilde{A}}$ as a direct factor. To see that $\tilde{H}_{\tilde{X}, \tilde{A}}$ is non-zero, note that $\tilde{T}_{\tilde{X}, \tilde{A}}$ is non-zero and acts faithfully on $\tilde{H}_{\tilde{X}, \tilde{A}}$. The second part can be proved in a similar way.

For the third part, we must show that the $A$-module $p_{\infty}/(P_{\infty}^2 + p_{\infty}^2)$ vanishes after tensoring with $E$. But this $A$-module is the cokernel of an inverse limit of maps whose cokernels are finite torsion $A$-modules of uniformly bounded cardinality (by the hypothesis of the theorem).

Lemma 4.22. (i) Suppose that for each $v \in R$, the characters $\chi_{v,1}, \ldots, \chi_{v,n}$ are pairwise distinct. Then for each minimal prime $Q \subset \Lambda$, $R_{\chi, p_\infty}^\infty/Q$ is an $\mathcal{O}$-flat domain of dimension $n[L^+: \mathbb{Q}] + q + n^2[T]$.

(ii) Suppose instead that for each $v \in R$, the characters $\chi_{v,1}, \ldots, \chi_{v,n}$ are all trivial, in which case we write $\chi = 1$. Then for each minimal prime $Q \subset \Lambda$, $R_{\chi, p_\infty}^\infty/Q$ is equidimensional of dimension $n[L^+: \mathbb{Q}] + q + n^2[T]$, and all of its minimal primes have characteristic zero. Moreover, each prime minimal over $\lambda$ contains a unique minimal prime.

Proof. For the first part of the lemma, we recall that by definition

$$\tilde{R}_{\tilde{S}^r_{\tilde{X}, T}} = \left( \bigotimes_{v \in S} R_v^\infty \right) \otimes_{\mathcal{O}} \left( \bigotimes_{v \in S(B)} R_v^\infty \right) \otimes_{\mathcal{O}} \left( \bigotimes_{v \in S_1} R_v^{\Lambda \otimes \Lambda} \right) \otimes_{\mathcal{O}} \left( \bigotimes_{v \in R} R_v^{\Lambda} \right),$$

and that $R_{\chi}^\infty$ is a power series ring over $\tilde{R}_{\tilde{S}^r_{\tilde{X}, T}}$ in $q$ variables. The ideal $\tilde{P}_{\operatorname{loc}} \subset \tilde{R}_{\tilde{S}^r_{\tilde{X}, T}}$ is the kernel of the homomorphism $\tilde{R}_{\tilde{S}^r_{\tilde{X}, T}} \to A$ which classifies the local restrictions $r_{p|G_{L_0}}$ for $v \in T$, and $P_{\infty}$ is the pullback of the ideal $\tilde{P}_{\operatorname{loc}}$ under the natural homomorphism

$$R_{\chi}^\infty \to \tilde{R}_{\tilde{S}^r_{\tilde{X}, T}} \to A.$$

As such, the extension $\Lambda \subset \tilde{A}$ having been fixed, the truth of the lemma depends only on these local restrictions.
We first reduce to the case that for \( v \in S(B) \), \( \tau_p|_{G_{\Lambda, v}} \) is in fact trivial. By hypothesis, these representations are unramified and \( \tau_p(Frob_v) = \alpha_v \cdot 1_n \), for \( \alpha_v \in 1 + m_\Lambda \) a suitable scalar. Choose for each \( v \in S(B) \) a lift \( \tilde{\alpha}_v \) to \( \tilde{\Lambda} \). Then there is an induced homomorphism \( f_{\tilde{\alpha}} : \tilde{R}^{\text{loc}}_{S, T} \to \tilde{R}^{\text{loc}}_{S, T} \) which classifies the universal lifting for \( v \in T - S(B) \), and the unramified twist of the universal lifting by \( \tilde{\alpha}_v \) for \( v \in S(B) \). Then \( f_{\tilde{\alpha}}(P^\text{loc}) \) is the kernel of the surjective homomorphism

\[
\tilde{R}^{\text{loc}}_{S, T} / \tilde{R}^{\text{st}}_v = \left( \bigotimes_{v \in S(B)} R_v^\Delta \otimes_{\Lambda} \tilde{\Lambda} \right) \otimes_{\O} \left( \bigotimes_{v \in S_a} R_v^\square / \lambda \right) \to A,
\]

call this \( P' \), and we get an isomorphism \( \tilde{R}^{\text{loc}}_{S, T, \tilde{p}^\text{loc}} \cong \tilde{R}^{\text{loc}}_{S, T, P'} \). We write

\[
C = \left( \bigotimes_{v \in S(B)} R_v^\Delta \otimes_{\Lambda} \tilde{\Lambda} \right) \otimes_{\O} \left( \bigotimes_{v \in S_a} R_v^\square \right),
\]

and \( P'_1 \) for the kernel of the map \( C \to A \). We write

\[
D = \left( \bigotimes_{v \in S(B)} R_v^{\text{st}} \right) \otimes_{\O} \left( \bigotimes_{v \in S} R_v^\square \right),
\]

and \( m_D \) for its maximal ideal. Thus \( P' = (P'_1, m_D) \subset C \otimes_{\O} D \).

For each minimal prime \( Q \subset \Lambda \), the ring \((C/Q)_{P'_1}\) is a formally smooth \((\Lambda/Q)_{\tilde{p}^\text{loc}}\)-algebra. To see this, we compute the relative tangent space \( (C/Q)_{P'_1} \) of the morphism \((\Lambda/Q)_{\tilde{p}^\text{loc}} \to (C/Q)_{P'_1}\). By Lemma \[3.12\] the \( E \)-dual of this tangent space is isomorphic to space of data of the following type: a choice for each \( v \) a pair \((\text{Fil}^*, \tau)\) of a lifting of \( \tau_{v[N]} \) to \( E[\epsilon] \) and a filtration preserved by it, such that the diagonal are the pushfowards of \((\psi^v_1, \ldots, \psi^v_n)\) via the homomorphism \( \Lambda/P \to E \), and for each \( v \in S_a \), an unramified lifting of \( \tau_{v[N]} \) to \( E[\epsilon] \). Arguing as in the proof of \[Ger\] Lemma 3.2.3 shows that this tangent space has dimension equal to \( n(n - 1)/2 |L^+ : Q| + n^2|S_B| + n^2|S_a| = \dim(C/Q)_{P'_1} - \dim(\Lambda/Q)_{\tilde{p}^\text{loc}}. \) It follows that \((C/Q)_{P'_1}\) is a formally smooth \((\Lambda/Q)_{\tilde{p}^\text{loc}}\)-algebra, and hence that this ring can be presented as a power series ring over \((\Lambda/Q)_{\tilde{p}^\text{loc}}. \)

On the other hand, there is an isomorphism

\[
\tilde{R}^{\text{loc}}_{S, T, P'_1} \cong C_{P'_1} \otimes_{\O} D.
\]

To see this, we construct the maps in either direction. First, recall that \( C_{P'_1} \otimes_{\O} D \) is the initial object in the category of complete Noetherian semi-local \( \O \)-algebras with \( \O \)-algebra maps from \( C_{P'_1} \) and \( D \). There are maps \( C_{P'_1} \to (C \otimes_{\O} D)_{P'} \) and \( D \to (C \otimes_{\O} D)_{P'} \) (note that \( P'_1 \) is the pullback of \( P' \) to \( C \) under the natural map \( C \to C \otimes_{\O} D \)). In the other direction, we note that there is a natural map \( C \otimes_{\O} D \to C_{P'_1} \otimes_{\O} D \), and the pullback of the maximal ideal under this map is \( P' \). This gives rise to a natural map \((C \otimes_{\O} D)_{P'} \to C_{P'_1} \otimes_{\O} D\). A similar argument shows that \( R_{\chi, P'^{\text{loc}}} \) is isomorphic to a power series ring over \( \tilde{R}^{\text{loc}}_{S, T, \tilde{p}^\text{loc}} \) in \( q \) variables.

To finish the proof of the first part of the lemma, it therefore suffices to show that for each minimal prime \( Q \subset \Lambda \), \( \tilde{R}^{\text{loc}}_{S, T, \tilde{p}^\text{loc}}/Q \) is an \( \O \)-flat domain of dimension \( n|L^+ : Q| + n^2|T| \), or equivalently that \( C_{P'_1}/Q \otimes_{\O} D \) is an \( \O \)-flat domain of this dimension.

Successively applying Proposition \[3.14\] and Proposition \[3.16\] we see that \( \text{Spec } C_{P'_1}/Q \otimes_{\O} D[1/\ell] \) is connected. On the other hand, the morphism \( R^{\text{loc}}_{S, T, (P'_1)[1/\ell]} \to R^{\text{loc}}_{S, T, P'[1/\ell]} \) is regular, being a localization of a regular morphism (see \[Mat89\] §32 for the definition and basic properties of this notion) and faithfully flat. Moreover, the ring \( R^{\text{loc}}_{S, T, (P'_1)[1/\ell]} \) is regular. To see this, note that the non-regular locus of \( R^{\text{loc}}_{S, T, (P'_1)[1/\ell]} \) is a closed set, and \( R^{\text{loc}}_{S, T, (P'[1/\ell]} \) is a Jacobson ring. By Lemma \[3.11\] and the proof of \[Kis09\] Lemma 3.4.12, this ring is formally smooth over \( K \) above the open subset of \( \Lambda[1/\ell] \) where for each \( 1 \leq i < j \leq n \), we have \( \psi^v_i \neq \psi^v_j \). This locus contains \( R^{\text{loc}}_{S, T, (P'_1)[1/\ell]} \), by hypothesis. It follows that \( R^{\text{loc}}_{S, T, P'/Q}[1/\ell] \) is regular with connected spectrum, and is therefore a domain.

For the second part of the lemma, we apply the following result, which generalizes \[Tay08\] Lemma 2.7, to \( R^{\text{loc}}_{T, (P\infty)} \).
Proposition 4.23. Let $A$ be an excellent local $O$-algebra. Let $X_1,\ldots,X_r$ be the distinct irreducible components of Spec $A$, endowed with their reduced subscheme structure. Suppose that $A$ satisfies the following hypotheses:

- For each $i$, $X_i$ is $O$-flat and $X_i \otimes_O k$ is generically reduced of dimension $d$.
- For each $i \neq j$, $X_i \otimes_O k$ and $X_j \otimes_O k$ have no irreducible components in common.

Let $B = \widehat{A}$. Then $B$ is equidimensional of dimension $d + 1$ and satisfies the following:

- Every minimal prime of $B$ has characteristic zero and is contained in a prime minimal over $\lambda$.
- Every prime of $B$ minimal over $\lambda$ contains a unique minimal prime of $B$.

Proof. We may suppose without loss of generality that $A$ is reduced. First note that $A$ is equidimensional of dimension $d + 1$, hence $B$ is equidimensional of dimension $d + 1$ by [Mat89 Corollary 31.5]. Let $\widehat{A}$ denote the normalization of $A$. Thus Spec $\widehat{A} = \bigcup \tilde{X}_i$, where $\tilde{X}_i$ denotes the normalization of $X_i$, and the union is disjoint.

Write $X_i \otimes_O k = \bigcup_j Y_{ij}$ as a union of distinct irreducible components. Now $X_i$ and $\tilde{X}_i$ are $O$-flat and equidimensional, hence $X_i \otimes_O k$ and $\tilde{X}_i \otimes_O k$ are equidimensional by [Mat89 Theorem 31.5], and $\tilde{X}_i \otimes_O k \rightarrow X_i \otimes_O k$ is finite. Let $\tilde{Y}_{ij}$ be the union of the components of $\tilde{X}_i \otimes_O k$ which map onto $Y_{ij}$.

We now argue as in the proof of [Tay08 Lemma 2.7]. Let $\tilde{\eta}_{ij}$ denote the generic point of $\tilde{Y}_{ij}$. We see that $\mathcal{O}_{\tilde{X}_i, \tilde{\eta}_{ij}} = \mathcal{O}_{X_i, \eta_{ij}}$, so there is a unique point of $\tilde{X}_i$ above $\eta_{ij}$, and $\tilde{Y}_{ij}$ is irreducible and dominates $Y_{ij}$.

Let $Q = \widehat{A}/A$. Then we have exact sequences

$$0 \rightarrow A \rightarrow \widehat{A} \rightarrow \tilde{Q} \rightarrow 0$$

and

$$0 \rightarrow B \rightarrow \prod_{i,y} \mathcal{O}_{\tilde{X}_i,y} \rightarrow \tilde{Q} \rightarrow 0,$$

where the product is taken over the finitely many closed points $y$ of $\tilde{X}_i$. Let $p_{i,y} = \ker(B \rightarrow \mathcal{O}_{\tilde{X}_i,y})$, a minimal prime of $B$. If $\varphi \subset B$ is any minimal prime, then $\mathcal{O}_{\varphi} = \varphi \cap A$ is a minimal prime of $A$, and hence $Q_{(\varphi)} = 0$ and there exists $f \in A - \mathcal{O}_{\varphi}$ such that $fQ = 0$. This implies that $f\tilde{Q} = 0$ and hence $\tilde{Q}_{(\varphi)} = 0$ and $B_{(\varphi)} = \prod_{i,y} (\mathcal{O}_{\tilde{X}_i,y})_{(\varphi)}$. Consequently $\varphi = p_{i,y}$ for a unique pair $(i,y)$, and the $p_{i,y}$ are the distinct minimal primes of $B$.

Similarly, if $\varphi \subset B$ is a prime above a minimal prime of $A \otimes_O k$ (which thus corresponds to the generic point of some $Y_{ij}$), then again $\tilde{Q}_{(\varphi)} = 0$, and since $B_{(\varphi)}$ is local we see that $\varphi \supset p_{i,y}$ for a unique pair $(i,y)$.

Let $\varphi$ be a prime of $B$ minimal over $(\lambda, p_{i,y})$. Then we claim that $\varphi$ is minimal over $\lambda$, which gives the first bullet point above. For otherwise, there exists a strict chain of inclusions

$$\varphi \supset q \supset \lambda B,$$

where $q \subset B$ is a prime. Then $q$ contains a prime $p_{j,z}$ for some $(i, y) \neq (j, z)$, and we have

$$\dim B/p_{i,y} = 1 + \dim B/(\lambda, p_{i,y}) = 1 + \dim B/\varphi + 1 + \dim B/q \leq \dim B/p_{j,z},$$

a contradiction, since $B$ is equidimensional.

Finally we come to the second bullet point. Let $\varphi \subset B$ be a prime minimal over $\lambda$. By the going-down theorem, $\varphi$ lies above a prime of $A$ minimal over $\lambda$. We have shown above that such a prime contains a unique minimal prime of $B$, as required. \qed
To see that $R_{\chi,\infty}^{\infty} \tensor \Lambda/Q$ satisfies the hypotheses of Proposition 4.23, it suffices to show that $R_{\chi,1}^{\infty} / Q$ satisfies the same conditions, since localizing at $\mathfrak{p}^\infty_{\infty}$ does not affect either point. We first argue as in the paragraphs following [BLGHT11, Lemma 3.3] to calculate the irreducible components of $R_{\chi,1}^{\infty} / Q$ (after possibly enlarging $K$, which is harmless). This gives the second bullet point above, once we have checked that the ring

$$\hat{\otimes}_{v \in S_1} R_{\chi,1}^{\infty} \tensor_{\Lambda} \hat{\Lambda}/Q$$

has irreducible spectrum for each minimal prime $Q \subset \Lambda$. This ring is $\mathcal{O}$-flat, modulo nilpotents, and the fibers of the morphism

$$f : \text{Spec} \hat{\otimes}_{v \in S_1} R_{\chi,1}^{\infty} \tensor_{\Lambda} \hat{\Lambda}/Q[1/l] \to \text{Spec} \hat{\Lambda}/Q[1/l]$$

are regular away from the closed subset of $\text{Spec} \hat{\Lambda}/Q[1/l]$ where for some $v \in S_1$ and $1 \leq i, j \leq n$ we have $\psi_i^v = \psi_j^v$ or $\psi_i^v = c \psi_j^v$. Write $V \subset \text{Spec} \hat{\Lambda}/Q[1/l]$ for the complement of this closed subset. Then $f_V$ is flat with irreducible fibers. It follows that $f^{-1}(V)$ is integral. On the other hand, the morphism

$$\text{Spec} \hat{\otimes}_{v \in S_1} R_{\chi,1}^{\infty} \tensor_{\Lambda} \hat{\Lambda}/Q[1/l] \to \text{Spec} \hat{\otimes}_{v \in S_1} R_{\chi,1}^{\infty} / Q[1/l]$$

is flat, so every irreducible component of the first space dominates a component of the second. This implies that $f^{-1}(V)$ is Zariski dense in the first space, which is therefore irreducible.

For the first bullet point, we must show that $R_{\chi,1}^{\infty} / Q, \lambda$ is generically reduced. Arguing as in the proof of [BLGHT11, Lemma 3.3], we see that it suffices to show that each of the rings $\hat{\otimes}_{v \in S_1} R_{\chi,1}^{\infty} \tensor_{\Lambda} \hat{\Lambda}/Q, \lambda$, $R_{\chi,1}^{\infty} / \mathfrak{p}^\infty_{\infty}$ and $R_{\chi,1}^{\infty} \tensor_{\Lambda} \hat{\Lambda}/Q, \lambda$ is generically reduced. This is follows from what we have already proved for each of these rings, except possibly for $\hat{\otimes}_{v \in S_1} R_{\chi,1}^{\infty} \tensor_{\Lambda} \hat{\Lambda}/Q, \lambda$. It follows from Lemma 3.12 that if $\eta_Q$ is the generic point of $\text{Spec} \Lambda/(Q, \lambda)$, then $\hat{\otimes}_{v \in S_1} R_{\chi,1}^{\infty} \tensor_{\Lambda} \kappa(\eta_Q)$ is reduced. By construction, there is a unique point $\tilde{\eta}_Q$ of $\text{Spec} \Lambda/(Q, \lambda)$ above $\eta_Q$, and the extension $\kappa(\tilde{\eta}_Q)/\kappa(\eta_Q)$ of residue fields is separable. It follows that $\hat{\otimes}_{v \in S_1} R_{\chi,1}^{\infty} \tensor_{\Lambda} \kappa(\tilde{\eta}_Q)$ is reduced, and hence $\hat{\otimes}_{v \in S_1} R_{\chi,1}^{\infty} \tensor_{\Lambda} \Lambda/(Q, \lambda)$ is generically reduced, as desired.

We now complete the proof of the theorem. We recall that we have constructed for each choice of $\chi$ a commutative diagram

$$\begin{array}{ccc}
S_{\infty} & \longrightarrow & A_{\chi,\infty} \\
\downarrow & & \downarrow \tilde{P}_{\chi} \\
R_{\chi,1}^{\infty} & \longrightarrow & B_{\chi,\infty} \\
& & \downarrow \tilde{R}_{\chi,1}^{\text{univ}} \\
\end{array}$$

and a module $M_{\chi,\infty}$ for $A_{\chi,\infty}$ which is free over $S_{\infty}$ of rank $s$. For varying $\chi$ these diagrams are identified modulo $\lambda$. The map $R_{\chi,1}^{\infty} \tensor \Lambda/Q \to B_{\chi,\infty} / \mathfrak{p}_{\infty}$ is surjective, and the map $A_{\chi,\infty} / \mathfrak{p}_{\infty} / Q \to B_{\chi,\infty} / \mathfrak{p}_{\infty}$ is surjective with nilpotent kernel. We can therefore identify $\text{Spec} A_{\chi,\infty} / \mathfrak{p}_{\infty}$ with a closed subspace of $\text{Spec} R_{\chi,1}^{\infty} / \mathfrak{p}_{\infty}$.

We now suppose either that $\chi = 1$ or that for each $v \in R$, the characters $\chi_{e,1}, \ldots, \chi_{e,n}$ are pairwise distinct. For any minimal prime $Q$ of $\Lambda$, $M_{\chi,\infty,1} / Q$ is a free $S_{\infty} / \mathfrak{p}_{\infty} / Q$-module and $\dim S_{\infty} / \mathfrak{p}_{\infty} / Q = \dim R_{\chi,1}^{\infty} / \mathfrak{p}_{\infty} / Q$, both of these rings being equidimensional. It follows that $\dim A_{\chi,\infty} / Q = S_{\infty} / \mathfrak{p}_{\infty} / Q$ and $\text{Supp}_{A_{\chi,\infty} / Q} M_{\chi,\infty} / Q$ is a union of irreducible components of $\text{Spec} R_{\chi,1}^{\infty} / Q$, which are necessarily all of characteristic zero.

Suppose that $\chi \neq 1$. Then the first part of Lemma 4.22 implies that $\text{Spec} R_{\chi,1}^{\infty} / Q$ is irreducible, and hence

$$\text{Supp}_{A_{\chi,\infty} / Q} M_{\chi,\infty} / Q = \text{Spec} A_{\chi,\infty} / Q = \text{Spec} R_{\chi,1}^{\infty} / Q.$$ 

Therefore $\text{Supp}_{A_{\chi,\infty} / Q} M_{\chi,\infty} / Q \subset \text{Spec} R_{\chi,1}^{\infty} / Q$ is a union of irreducible components of characteristic zero, and

$$\text{Supp}_{A_{\chi,\infty} / Q} M_{\chi,\infty} / Q \cap \text{Spec} R_{\chi,1}^{\infty} / (Q, \lambda) = \text{Spec} R_{\chi,1}^{\infty} / (Q, \lambda).$$ 

By the second part of Lemma 4.22, this can be true only if we in fact have

$$\text{Supp}_{A_{\chi,\infty} / Q} M_{\chi,\infty} / Q = \text{Spec} R_{\chi,1}^{\infty} / Q$$

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for each minimal prime $Q \subset \Lambda$. Dividing out further by the ideal $a$, we see that
\[
\text{Supp} \tilde{p}_{S_1, a}/Q \tilde{H}_{1, a}/Q = \text{Spec} \tilde{p}_{S_1, a}/Q,
\]
and hence $\tilde{H}_{1, a}$ is nearly faithful $\tilde{p}_{S_1, a}$-module. This concludes the proof.

## 5 Taylor-Wiles systems

Let $k$ be a finite field of characteristic $l > 3$, $A = k[[T]]$, and $E = \text{Frac} A$. If $M$ is an $A$-module and $x \in M$ is killed by a power of $T$, we will call the least integer $n \geq 0$ such that $T^n x = 0$ the order of $x$.

Let $\Gamma = \Delta \rtimes \{1, c\}$ be a profinite group. We suppose given a continuous representation $r : \Gamma \to G_n(A)$ such that $\Delta = r^{-1}(G_n^\sigma(A))$. This section is devoted to some group theoretical results about such representations, which will allow us to verify the hypotheses of Theorem 4.17 in certain situations. We establish the following running hypotheses:

(i) $r \otimes_A E$ is absolutely irreducible.

(ii) $\Gamma$ is Schur, and $\pi \mid \Delta$ is primitive (i.e. not induced from any proper subgroup of $\Delta$).

(iii) There exists $\sigma_0 \in \Delta$ such that $r(\sigma_0) \in \text{GL}_n(A)$ is regular semisimple, and its eigenvalues lie in $A^\times$ and do not satisfy any non-trivial $\mathbb{Z}$-linear relation in $A^\times$. (It may be helpful to recall that there is an isomorphism $A^\times \cong k^\times \times \prod_{i=1}^\infty \mathbb{Z}_l$.)

(iv) The integer $n$ is not divisible by $l$.

(v) We have $r(c) = (J, \mu)j$ where $J$ is a symmetric matrix. Equivalently (cf. Lemma 3.1), $\nu \circ r(c) = -1$.

**Proposition 5.1.** For each subgroup $N \subset \Delta$ of finite index, $r \mid_N$ is absolutely irreducible.

**Proof.** Suppose not. Then $\sigma_0^N \in N$ for some $a \geq 1$, and hence the representation $r \mid_N \otimes_A E$ is multiplicity-free (being already semisimple). Let $\rho \subset r \mid_N \otimes_A E$ be a simple subrepresentation. By Clifford theory, it follows that the action of $N$ on $\rho$ extends to an action of a subgroup $N' \subset \Delta$ and that there is an isomorphism $r \mid_\Delta \cong \text{Ind}^\Delta_N \rho$ over $E$. Then $\pi \mid_\Delta \cong \text{Ind}^\Delta_N \pi$, both representations being semisimple. Since we have assumed that $\pi \mid_\Delta$ is primitive, it follows that $N' = \Delta$ and $\rho = r \mid_N \otimes_A E$, and hence this representation is irreducible. Enlarging the field $E$ does not affect our hypotheses, so we see that the representation is even absolutely irreducible. \( \square \)

**Proposition 5.2.** Let $G = r \mid_\Delta(\Delta) \subset \text{GL}_n(A)$, and let $Z$ denote its center. There exists a closed subfield $K \subset E$ and a descent $H$ of $\text{PGL}_n$ to $K$ such that $G/Z$ is identified with an open compact subgroup of $H(K)$.

**Proof.** By [Pin95], Theorem 0.7, it suffices to show that $G/Z$ is Zariski dense in $\text{PGL}_n$, viewed as algebraic group over $E$. Write $H'$ for the connected component of the Zariski closure; this is reductive. Since $\sigma_0 \in \Delta$, $H'$ contains a maximal torus of $\text{PGL}_n$. By the previous proposition, $H'$ has trivial center. This implies that $H' = \text{PGL}_n$. \( \square \)

**Lemma 3.3.** Let $\Gamma' \subset \Gamma$ be the kernel of a continuous homomorphism $\Gamma \to \mathbb{Z}_l$, and let $G' \subset r \mid_\Delta(\Gamma' \cap \Delta)$ be an open subgroup which is normalized by $c$. Then:

(i) The image of $\Gamma' \cap \Delta$ in $G/Z$ has finite index.

(ii) There exists an integer $K_0$, depending only on $G'$, such that for any element $x \in \text{ad}^0 r \otimes_A A/T^m$ of exact order $a$, we have $T^{m-a+K_0} \text{ad}^0 r \otimes_A A/T^m \subset A[G'] \cdot x$. In particular, we have
\[
T^{K_0} H^0(G', \text{ad}^0 r \otimes_A A/T^m) = 0
\]
for all $m \geq 0$. 

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(iii) There exists an integer $K_1$, depending only on $G'$, such that
\[ T^{K_1} H^1(G', \text{ad}^0 r \otimes_A A/T^m) = 0 \]
for all $m \geq 0$.

Proof. The first part follows from [Pin98, Theorem 0.2, (c)] and [Ser06, Ch. III, §9]. (We note that since $l$ is coprime to $n$, the covering $SL_n \to PGL_n$ is separable.) The second part is elementary, using that the adjoint representation of $PGL_n$ is irreducible.

For the proof of third part, we will use only that the image of $G'$ in $H(K)$ is an open compact subgroup which is normalized by $c$. We can therefore assume without loss of generality that $G' = G$. Let $[\phi] \in H^1(G, \text{ad}^0 r \otimes_A A/T^m)\cong$ be a cohomology class. Since $l \neq 2$, we can assume that $\phi^c = -\phi$, where by definition we have
\[ \phi^c(\sigma) = \text{ad}(c)\phi(\sigma^c) = -J^t\phi(\sigma^c)J^{-1}. \]
For $\sigma \in G$, we have $\sigma^c = J^t\sigma^{-1}J^{-1}$ and $\phi(\sigma^c) = J^t\phi(\sigma)J^{-1}$. In the course of the proof, we will repeatedly use the following observation: let $v \in (A/T^m)^N$, and $M \in M_{N\times N}(A)$. Suppose that $Mv = 0$. Then $\det(M)v = 0$.

We can view $\phi$ as attached to a representation $r_\phi : G \to \text{GL}_n(A \oplus \epsilon A/T^m)$ via the formula $r_\phi(\sigma) = (1 + \epsilon \phi(\sigma))r(\sigma)$. (Here and in the rest of the proof we write $r$ abusively for the induced representation of $G = r(\Delta)$.) The cocycle $\phi$ represents the trivial cohomology class if and only if this representation is $1 + \epsilon M_{n\times n}(A/T^m)$-conjugate to $r$. For each $\sigma \in G$, we therefore have
\[ \text{tr} r_\phi(\sigma^{-1}) = \text{tr} r(\sigma^{-1}) + \epsilon \text{tr} \phi(\sigma^{-1})r(\sigma^{-1}), \]
\[ \text{tr} r_\phi(\sigma) = \text{tr} r(\sigma) + \epsilon \phi(\sigma^c) \text{tr} r(\sigma^c) = \text{tr} r(\sigma^c) - \epsilon \text{tr} \phi(\sigma^{-1})r(\sigma^{-1}). \]
Thus if $\sigma^c$ and $\sigma^{-1}$ are $G$-conjugate, then $\text{tr} r_\phi(\sigma^{-1}) = \text{tr} r(\sigma^{-1})$. Let $X$ denote the closure in $G$ of the set of such $\sigma$. We note that for $\sigma \in X$, we have $\text{tr} r_\phi(\sigma^m) = \text{tr} r(\sigma^m)$ for each $m \in \mathbb{Z}$. In particular, the three lowest degree coefficients of the characteristic polynomial of $r_\phi(\sigma)$ agree with those of $r(\sigma)$, since these coefficients can be expressed in terms of $\text{tr} r(\sigma)$, $\text{tr} r(\sigma^2)$ and $\text{tr} r(\sigma^3)$, at least when the characteristic is not $2$ or $3$.

We claim that $X$ has positive Haar measure in $G$. Let $\mathfrak{h} = \text{Lie} H$. We have an automorphism $\theta$ of $H$ and of $\mathfrak{h}$ induced by that of $\text{ad}(c)$ on $G$, by [Pin98, Corollary 0.3]. Since $K$ is infinite, we can choose a regular semisimple element $x$ of $\mathfrak{h}^{\theta = -1}$; then the centralizer of $x$ in $H$ is a maximal torus $T$ on which $\theta$ acts by $h \mapsto h^{-1}$ (cf. [Lev97], Lemma 2.4). This uses that the matrix $J$ is symmetric. The inverse image of $(T \cap G/Z)^G$ in $G$ will have positive measure if $(T \cap G/Z)^G$ has positive measure in $G/Z$, and if $g$ lies in this inverse image then $g^\epsilon$ and $g^{-1}$ are $G$-conjugate. To prove the claim it suffices to show that $(T \cap G/Z)^G$ has positive measure in $G/Z$, and this is true since $(T \cap G/Z)^G$ contains an open subset of $G/Z$, by [Ser06, Ch. III, §9].

For any proper Zariski closed subset $C \subset PGL_n$ over $E$, $C(E) \cap G/Z$ has Haar measure zero. This can be deduced easily from the argument of [Lay93] §2, Lemma 2. In particular, given countably many Zariski-closed subsets $C_1, C_2, \ldots$ of $PGL_n$, we can find elements of $X$ whose image in $PGL_n(E)$ is not contained in any $C_i(E), i \geq 1$. Using this observation, we choose $\sigma \in X$ whose eigenvalues $\lambda_1, \ldots, \lambda_n$ satisfy the following conditions:

(i) For each sequence of integers $r_1 = 0 < r_2 < \cdots < r_n$,
\[ \det(\lambda_i^{r_j})_{i=1,\ldots,n,j=1,\ldots,n} \neq 0. \]
(ii) Write $M = (r_j)$. For each sequence of integers $r_1 = 0 < r_2 < \cdots < r_N$, the determinant of the matrix
\[
\begin{pmatrix}
1 & & & 1 \\
\lambda_1 \lambda_2 & (\lambda_1 \lambda_3)^{r_2} & \cdots & (\lambda_{n-1} \lambda_n)^{r_2} \\
\lambda_1 \lambda_2 & (\lambda_1 \lambda_3)^{r_3} & \cdots & (\lambda_{n-1} \lambda_n)^{r_3} \\
\cdots & \cdots & \cdots & \cdots \\
\lambda_1 \lambda_2 & \cdots & (\lambda_1 \lambda_3)^{r_M} & (\lambda_{n-1} \lambda_n)^{r_M}
\end{pmatrix}
\]
is not zero.

(iii) Write instead $M = \binom{n}{3}$. For each sequence of integers $r_1 = 0 < r_2 < \cdots < r_M$, the determinant of the matrix

$$
\begin{pmatrix}
1 & 1 & \ldots & 1 \\
(\lambda_1 \lambda_2 \lambda_3)^{r_2} & (\lambda_1 \lambda_2 \lambda_4)^{r_2} & \ldots & (\lambda_{n-2} \lambda_{n-1} \lambda_n)^{r_2} \\
(\lambda_1 \lambda_2 \lambda_3)^{r_3} & (\lambda_1 \lambda_2 \lambda_4)^{r_3} & \ldots & (\lambda_{n-2} \lambda_{n-1} \lambda_n)^{r_3} \\
\vdots & \vdots & \ddots & \vdots \\
(\lambda_1 \lambda_2 \lambda_3)^{r_M} & (\lambda_1 \lambda_2 \lambda_4)^{r_M} & \ldots & (\lambda_{n-2} \lambda_{n-1} \lambda_n)^{r_M}
\end{pmatrix}
$$

is not zero.

After possibly enlarging $E$ and replacing $\phi$ by $T^N \phi$ for some $N$ depending only on $\sigma$, we may assume that $\sigma$ is diagonal in the standard basis of $A^n$. For each integer $t \geq 0$ we have

$$
\text{tr} \ r_\phi(\sigma^t) - \text{tr} \ r(\sigma^t) = \epsilon \sum_{i=1}^{n} \lambda_i^{t-1} \phi(\sigma)_{i,i} = 0.
$$

Multiplying $\phi$ by the determinant of the matrix

$$
\begin{pmatrix}
1 & 1 & \ldots & 1 \\
\lambda_1 & \lambda_2 & \ldots & \lambda_n \\
\lambda_1^{l} & \lambda_2^{l} & \ldots & \lambda_n^{l} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{(n-1)l} & \lambda_2^{(n-1)l} & \ldots & \lambda_n^{(n-1)l}
\end{pmatrix}
$$

we can thus suppose that $\phi(\sigma)_{i,i} = 0$ for each $i$. Now after multiplying $\phi$ by the element $\lambda_i/\lambda_j - 1$ with highest valuation, $i \neq j$, we can alter $\phi$ by a coboundary to assume that $\phi(\sigma) = 0$ and hence $r_\phi(\sigma) = r(\sigma)$. To be precise, for any $y \in \text{ad} \ r \otimes_A M$, we have $(\sigma y \sigma^{-1} - y)_{i,j} = (\lambda_i/\lambda_j - 1)y_{i,j}$, so after scaling $\phi$ we can find $y$ with $(\sigma y \sigma^{-1} - y) = \phi(\sigma)$. Since $X$ has positive measure we can find integers $r_1 = 0 < r_2 < \cdots < r_\binom{n}{3}$ such that $Y = X \cap \sigma^{-r_2} X \cap \cdots \cap \sigma^{-r_\binom{n}{3}} X$ has positive measure. If $\gamma \in Y$, then for each $M = 1, \ldots, n$ we have

$$
\text{tr} \ r_\phi(\sigma^{-r_M} \gamma) = \sum_{i=1}^{n} \lambda_i^{r_M} r_\phi(\gamma)_{i,i} \in A.
$$

Thus after multiplying $\phi$ by the determinant of the matrix

$$
\begin{pmatrix}
1 & 1 & \ldots & 1 \\
\lambda_1^{r_2} & \lambda_2^{r_2} & \ldots & \lambda_n^{r_2} \\
\lambda_1^{r_3} & \lambda_2^{r_3} & \ldots & \lambda_n^{r_3} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{r_n} & \lambda_2^{r_n} & \ldots & \lambda_n^{r_n}
\end{pmatrix}
$$

the diagonal entries of $r_\phi(\gamma)$ must lie in $A$.

Now since $Y$ has positive measure, we can choose $\tau \in Y$ satisfying the following conditions:

(i) Write $r(\tau) = x$. For all integers $1 \leq i, j \leq n$, $x_{i,j}$ is not zero.

(ii) For all integers $1 \leq i < j < k \leq n$, the determinant of the matrix

$$
\begin{pmatrix}
x_{i,j} & x_{i,i} \\
x_{k,k} x_{i,j} - x_{i,k} x_{k,j} & x_{j,i} x_{k,k} - x_{k,i} x_{j,k}
\end{pmatrix}
$$

is not zero.
(iii) For each sequence \( s_1 = 0 < s_2 < \cdots < s_n \), and for each integer \( t = 1, \ldots, n \) define a matrix \( A(t)_{i,j} = r_{s_i s_j} \). Then \( \det A(t) \neq 0 \).

We claim that after changing \( \phi \) by a coboundary and multiplying by a power of \( T \) depending only on \( \sigma \) and \( \tau \), we shall have \( r_\phi(\tau) = r(\tau) \). To see this, first note that \( r_\phi(\tau)_{i,i} = r(\tau)_{i,i} \) for each \( i = 1, \ldots, n \). After multiplying \( \phi \) by the element \( r(\tau)_{i,j+1} \) with largest valuation and scaling the basis elements of \( A^m \) by elements of \( 1 + \epsilon A/T^m \) (equivalently, changing \( \phi \) by a coboundary), we can assume that \( r_\phi(\tau)_{j,j+1} = r(\tau)_{j,j+1} \) for each \( j = 2, \ldots, n \) and \( r_\phi(\sigma) = r(\sigma) \).

We now use the fact that the first three coefficients of the characteristic polynomials of \( r_\phi(\tau) \) and \( r(\tau) \) agree. Let us treat e.g. the case of the third coefficient. This is the sum, up to signs, of the determinants of the 3 × 3 submatrices obtained by fixing 1 ≤ \( i < j < k \leq n \) and taking the intersection of the \( i, j, k \) rows and the \( i, j, k \) columns. Comparing these determinants for \( r(\tau) \) and \( r(\sigma \tau) \), we see that they differ by \((\lambda_1 \lambda_2 \lambda_3)^m\). Thus, multiplying \( \phi \) by the determinant of the matrix

\[
\begin{pmatrix}
1 & (\lambda_1 \lambda_2 \lambda_3)^{r_1} & \cdots & (\lambda_n - 1 \lambda_n - 1 \lambda_n)^{r_1} \\
(\lambda_1 \lambda_2 \lambda_3)^{r_1} & (\lambda_1 \lambda_2 \lambda_4)^{r_2} & \cdots & (\lambda_n - 2 \lambda_n - 1 \lambda_n)^{r_2} \\
\vdots & \vdots & \ddots & \vdots \\
(\lambda_1 \lambda_2 \lambda_3)^{r_{(n-1)}} & (\lambda_1 \lambda_2 \lambda_4)^{r_{(n-1)}} & \cdots & (\lambda_n - 2 \lambda_n - 1 \lambda_n)^{r_{(n-1)}}
\end{pmatrix},
\]

the determinants of these 3 × 3 submatrices agree for \( r(\tau) \) and \( r_\phi(\tau) \). Multiplying by the determinant of a similar \( \binom{n}{2} \times \binom{n}{2} \) matrix, we obtain the same result for the 2 × 2 submatrices obtained by deleting all but the \( i, j \) rows and columns for fixed \( i \neq j \).

We now show by induction on \( |i - j| \) that, modifying \( \phi \) at each step in a way depending only on \( r(\tau) \), we can assume that \( r(\tau)_{i,j} = r_\phi(\tau)_{i,j} \) for each \( i, j \). The cases of \( i = j \) and \( i = j - 1 \) have already been solved. To conserve notation let us temporarily write \( r(\tau) = x \) and \( r_\phi(\tau) = x + \epsilon X \). To take care of the remaining case when \( j = i + 1 \), note that determinants

\[
\det \begin{pmatrix}
x_{i,i} & x_{i,j} \\
x_{j,i} & x_{j,j}
\end{pmatrix} = x_{i,i} x_{j,j} - x_{i,j} x_{j,i},
\]

\[
\det \begin{pmatrix}
x_{i,i} & x_{i,j} + \epsilon x_{j,i} \\
x_{j,i} & x_{j,j}
\end{pmatrix} = x_{i,i} x_{j,j} - x_{i,j} x_{j,i} - \epsilon x_{j,i} x_{i,j}
\]

have been shown to be equal; multiplying \( \phi \) by \( x_{i,j} \) kills \( X_{i,j} \).

For the induction step, fix \( i < k < j \) and consider the determinants

\[
\det \begin{pmatrix}
x_{i,i} & x_{i,j} \\
x_{j,i} & x_{j,j}
\end{pmatrix} = \det \begin{pmatrix}
x_{i,i} & x_{i,j} + \epsilon x_{j,i} \\
x_{j,i} + \epsilon x_{j,i} & x_{j,j}
\end{pmatrix},
\]

\[
\det \begin{pmatrix}
x_{i,i} & x_{i,k} & x_{i,j} \\
x_{k,i} & x_{k,k} & x_{k,j} \\
x_{j,i} & x_{j,k} & x_{j,j}
\end{pmatrix} = \det \begin{pmatrix}
x_{i,i} & x_{i,k} & x_{i,j} + \epsilon x_{i,j} \\
x_{k,i} & x_{k,k} & x_{k,j} \\
x_{j,i} + \epsilon x_{j,i} & x_{j,k} & x_{j,j}
\end{pmatrix},
\]

resulting in equations

\[
x_{i,j} X_{j,i} + x_{j,i} X_{i,j} = 0
\]

and

\[
(x_{k,k} x_{i,j} - x_{i,k} x_{k,j}) X_{j,i} + (x_{j,i} x_{k,k} - x_{k,i} x_{j,k}) X_{i,j} = 0.
\]

Multiplying \( \phi \) by the determinant of the matrix

\[
\begin{pmatrix}
x_{i,j} & x_{j,i} \\
x_{k,k} x_{i,j} - x_{i,k} x_{k,j} & x_{j,i} x_{k,k} - x_{k,i} x_{j,k}
\end{pmatrix}
\]

completes the induction step.
Since $Y$ has positive measure, we can find integers $s_1 = 0 < s_2 < \cdots < s_n$ such that $Z = Y \cap \tau^{-s_2}Y \cap \cdots \cap \tau^{-s_n}Y$ has positive measure. If $\zeta \in Z$ then for each $M = 1, \ldots, n$ we have $\tau^{s_M} \zeta \in Y$ and hence for each $i = 1, \ldots, n$

$$r_{\phi}(\tau^{s_M} \zeta)_{i,i} = \sum_{k=1}^{n} r(\tau^{s_M})_{i,k} r_{\phi}(\zeta)_{k,i} = r(\tau^{s_M} \zeta)_{i,i} = \sum_{k=1}^{n} r(\tau^{s_M})_{i,k} r(\zeta)_{k,i}.$$

After multiplying $\phi$ by the quantity $\det A(i)$ with largest valuation, we have that $r_{\phi}(\zeta) = r(\zeta)$ for all $\zeta \in Z$, and hence for all $\zeta$ in the closed subgroup generated by $Z$. Being of positive measure, this subgroup is open of finite index in $G$, and contains an open normal subgroup $N$. We note that the definition of $Z$ depends only on $G$ and not on $\phi$, and $N$ can therefore be chosen to depend only on $G$ and not on $\phi$. We now use the inflation restriction exact sequence for $N$:

$$0 \to H^1(G/N, (\text{ad}^0 r \otimes_A A/T^m)^N) \to H^1(G, \text{ad}^0 r \otimes_A A/T^m) \to H^1(N, \text{ad}^0 r \otimes_A A/T^m).$$

We have shown that for any cocycle $\phi$ for $G$ with $\phi^c = -\phi$, the restriction of the cohomology class of $\phi$ to $N$ is annihilated by $T^{K_2}$ for some integer $K_2$ depending on $G$ but not on $\phi$. By the second part of the lemma applied to $G' = N$, the first group in this sequence is annihilated by some $T^{K_2}$ depending only on $N$. The third part of the lemma now follows on taking $K_1 = K_0 + K_2$. \hfill \Box

5.1 Galois theory

We now return to the notation of the beginning of §2. Thus $F$ is an imaginary CM field with totally real subfield $F^+$. We fix a finite set of places $S$ of $F^+$ which split in $F$, and write $F(S)$ for the maximal extension of $F$ unramified outside $S$. We write $G_{F+,S} = \text{Gal}(F(S)/F^+)$ and $G_{F,S} \subset G_{F+,S}$ for the subset of elements fixing $F$. For each $v \in S$ we choose a place $\tilde{v}$ of $F$ above it, and write $\tilde{S}$ for the set of these places. We choose a complex conjugation $c \in G_{F+,S}$. We suppose that $S$ contains all places dividing $l$.

We now suppose given a representation $r : G_{F+,S} \to \mathcal{G}_n(A)$ satisfying the hypotheses of the previous section, with $\Gamma = G_{F+,S}$ and $\Delta = G_{F,S}$. We suppose further that:

- $\zeta \notin F$ and $\tau|_{G_{F+(\zeta)}}$ is Schur.
- The image of $\tau|_{G_{F+(\zeta)}}$ has no non-trivial quotients of $l$-power order. This will be the case if, for example, the irreducible constituents of $\tau|_{G_{F+(\zeta)}}$ are adequate in the sense of [Tho12].

We will be concerned with a slight variant of the cohomology groups defined in [CHT08]. Fix a $\Lambda$-algebra structure on $A$ and a deformation problem

$$\mathcal{S} = \left( F/F^+, S, \tilde{S}, \Lambda, \tau, \chi, \{D_v\}_{v \in S} \right),$$

such that $r$ is of type $\mathcal{S}$. We fix $T = S$, $\tilde{T} = \tilde{S}$. (Thus $T$ denotes both a set of places of $F^+$ and an element of the base ring $A$, but we hope that this will not cause confusion.) Fix also a choice of Taylor-Wiles data $(Q_N, \tilde{Q}_N, \{\tilde{\pi}_v\}_{v \in Q_N})$ of order $q$ and level $N$. (We allow the case $q = 0$.) If $n_v$ is the multiplicity of $\tilde{\pi}_v$ as an eigenvalue of $\tau(\text{Frob}_v)$ then we suppose $n_v$ coprime to $l$. We write $S_N = S \cup Q_N$, $\tilde{S}_N = \tilde{S} \cup \tilde{Q}_N$. This induces an auxiliary deformation problem

$$\mathcal{S}_N = \left( F/F^+, S_N, \tilde{S}_N, \Lambda, \tau, \chi, \{D_v\}_{v \in S} \cup \{D_{T}^{TW}(\tilde{\pi}_v)\}_{v \in Q_N} \right).$$

Then for each integer $m \geq 1$ and place $v \in Q_N$ there are associated submodules $L_{v,m} \subset H^1(G_{F_v}, \text{ad}^0 r \otimes_A A/T^m)$. To define these, we note that the choice of Taylor-Wiles data induces a direct sum decomposition $r|_{G_{F_v}} = s_v \oplus \psi_v$. We have a natural map

$$H^1(G_{F_v}, \text{ad}^0 r \otimes_A A/T^m) \to H^1(I_{\tilde{v}}, \text{ad}^0 r \otimes_A A/T^m)^{G_{F_v}},$$

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(\psi_v) \subset \text{ad } \psi_v denotes the submodule of diagonal matrices. We write \( L_{v,m} \) for the pre-image in the cochain group \( C^1(G_{F_v}, \text{ad } r \otimes A/T^m) \) of \( L_{v,m} \). Then we define

\[
C^i_{SN,T}(G_{F^+,S_N}, \text{ad } r \otimes A/T^m) = 
\]

\[
C^i(G_{F^+,S_N}, \text{ad } r \otimes A/T^m) \oplus \bigoplus_{v \in S_N} C^{i-1}(G_{F_v}, \text{ad } r \otimes A/T^m)/M^{i-1}_{v,m},
\]

where \( M^{i}_{v,m} = 0 \) unless \( v \in Q_N \) and \( i = 0 \), in which case we set \( M^{0}_{v,m} = C^0(G_{F_v}, \text{ad } r \otimes A/T^m) \) or \( v \in Q_N \) and \( i = 1 \), in which case we set \( M^{1}_{v,m} = L^1_{v,m} \). The boundary map is given by the formula

\[
\partial(\phi, (\psi_v)_{v \in S_N}) = (\partial\phi, (\psi_v - \partial\psi_v)_{v \in S_N}).
\]

The groups \( H^*_{SN,T}(G_{F^+,S_N}, \text{ad } r \otimes A/T^m) \) are then by definition the cohomology groups of this complex.

We are also given dual Selmer conditions \( L^{\perp}_{v,m} \subset H^1(G_{F_v}, \text{ad } r(1) \otimes A/T^m) \) for \( v \in Q_N \), defined to be the annihilator of \( L_{v,m} \) under the local duality pairing. We define a group

\[
H^1_{S_N,T}(G_{F^+,S_N}, \text{ad } r(1) \otimes A/T^m) = 
\]

\[
\ker H^1(G_{F^+,S_N}, \text{ad } r(1) \otimes A/T^m) \rightarrow \bigoplus_{v \in Q_N} H^1(G_{F_v}, \text{ad } r(1) \otimes A/T^m)/L^1_{v,m}.
\]

Finally, we write

\[
H^1_{S_N,T}(G_{F^+,S_N}, \text{ad } r \otimes A E/A) = \lim_{m} H^1_{S_N,T}(G_{F^+,S_N}, \text{ad } r \otimes A/T^m),
\]

and similarly for \( H^1_{S_N,T}(G_{F^+,S_N}, \text{ad } r(1) \otimes A E/A) \).

**Proposition 5.4.** (i) For each \( m \geq 0 \), we have

\[
H^3_{S_N,T}(G_{F^+,S_N}, \text{ad } r \otimes A/T^m) = H^3_{S_N,T}(G_{F^+,S_N}, \text{ad } r \otimes A E/A)[T^m].
\]

(ii) For each \( m \geq 0 \), we have

\[
H^1_{S_N,T}(G_{F^+,S_N}, \text{ad } r(1) \otimes A/T^m) = H^1_{S_N,T}(G_{F^+,S_N}, \text{ad } r(1) \otimes A E/A)[T^m].
\]

(iii) For each \( m \geq 0 \), we have

\[
|H^1_{S_N,T}(G_{F^+,S_N}, \text{ad } r \otimes A/T^m)| = 
\]

\[
|H^1_{S_N,T}(G_{F^+,S_N}, \text{ad } r(1) \otimes A/T^m)| \times |A/T^m|^{-u(n-1)/2[F^+ : \mathbb{Q}]},
\]

Proof. Write

\[ M_m = \text{ad } r \otimes A/T^m, M = \lim_{m} M_m = \text{ad } r \otimes A E/A. \]

We have exact sequences for every \( m' \geq m \)

\[
0 \longrightarrow M_m \longrightarrow M_{m'} \longrightarrow M_{m'-m} \longrightarrow 0.
\]

Since \( \tau \) is Schur, we have \( H^0(G_{F^+,S}, M_1) = 0 \) and hence \( H^0(G_{F^+,S}, M_m) = 0 \) for each \( m \geq 1 \). It follows that we have exact sequences

\[
0 \longrightarrow H^1(G_{F^+,S}, M_m) \longrightarrow H^1(G_{F^+,S}, M_{m'}) \longrightarrow H^1(G_{F^+,S}, M_{m'-m})
\]

for each \( m' \geq m \). Since the multiplication by \( T^m \) map on \( M_{m'} \) factors \( M_{m'} \rightarrow M_{m'-m} \hookrightarrow M_{m'} \) and this last inclusion also induces an injection on \( H^1 \), we find that we can identify

\[
H^1(G_{F^+,S}, M_m) = H^1(G_{F^+,S}, M_{m'})[T^m].
\]
Lemma 5.5. (i) There exists an integer $K_0 > 0$, not depending on $N$ or $m$, such that for every $N, m \geq 1$,

$$T^{K_0} H^1(G_{F^+}, \mathcal{M}_m(1)) / L^\perp_{v,m} \longrightarrow H^1(G_{F^+}, \mathcal{M}_{m'}(1)) / L^\perp_{v,m'}$$

are injective.

For the first part, we note (cf. the discussion after [CHT08, Definition 2.2.7]) that $H^1_{S_N,T}(G_{F^+,S}, \mathcal{M}_m)$ fits into an exact sequence

$$0 \longrightarrow \oplus_{v \in T} H^0(G_{F^+}, \mathcal{M}_m) \longrightarrow H^1_{S_N,T}(G_{F^+,S}, \mathcal{M}_m) \longrightarrow H^1(G_{F^+,S}, \mathcal{M}_m).$$

It follows that we have inclusions for every $m' \geq m$

$$H^1_{S_N,T}(G_{F^+,S}, \mathcal{M}_m) \subset H^1_{S_N,T}(G_{F^+,S}, \mathcal{M}_{m'}) [T^m].$$

We show equality. Suppose that $(\phi, (\alpha_v)_{v \in S})$ represents a cohomology class in the group on the right. By the above reasoning, we can assume that $\phi \in Z^1(G_{F^+,S}, \mathcal{M}_m)$. Then we have

$$T^m[(\phi, (\alpha_v)_{v \in S})] = [(0, (T^m \alpha_v)_{v \in S})] = 0,$$

and so there exists $Q \in \mathcal{M}_{m'}$ such that $(0, (T^m \alpha_v)_{v \in S}) = \partial Q$. Thus $Q \in H^0(G_{F^+,S}, \mathcal{M}_{m'}) = 0$ and hence $T^m \alpha_v = 0$, as desired.

The third part is proved exactly as in [CHT08, Lemma 2.3.4].

We define certain field extensions that will be used below. Let $F_{\infty}^+$ be the extension of $F^+$ obtained by adjoining all $l$-power roots of unity. For $m, N \geq 1$, let $L_m,N$ be the extension of $F^+((\zeta_N))$ cut out by the representation

$$r \mod T^m : G_{F^+,S} \to \mathcal{G}_m(A/T^m).$$

Let $L_{\infty}$ be the extension of $F_{\infty}^+$ cut out by $r$.

Lemma 5.5. (i) There exists an integer $K_0 > 0$, not depending on $N$ or $m$, such that for every $N, m \geq 1$,

$$T^{K_0} H^1(F^+((\zeta_N))/F^+, (\text{ad } r(1) \otimes A/T^m)^{G_{F^+((\zeta_N))}}) = 0.$$

(ii) There exists an integer $K_1 > 0$, not depending on $N$ or $m$, such that for every $N, m \geq 1$,

$$T^{K_1} H^1(\text{Gal}(L_m,N/F^+((\zeta_N))), \text{ad}^0 r(1) \otimes A/T^m) = 0.$$

(iii) There exists an integer $K_2 > 0$, not depending on $N$ or $m$, such that for every $N, m \geq 1$ and for any $G_{F^+((\zeta_N))}$-submodule $M \subset \text{ad}^0 r \otimes A/T^m$ containing an element of exact order $m - a$, we have $T^{a+K_2} \text{ad}^0 r \otimes A/T^m \subset M$.

Proof. For the first and third parts, apply the second part of Lemma 5.3 and the fact that $G_{F^+((\zeta_N))}$ acts on the diagonal matrices through the non-trivial character $\delta_{F/F^+}$. For the second part, we use the inflation-restriction exact sequence. First, we have

$$0 \to H^1(\text{Gal}(F_{\infty}^+/F^+((\zeta_N))), (\text{ad}^0 r(1) \otimes A/T^m)^{G_{F^+((\zeta_N))}}) \to H^1(\text{Gal}(L_{\infty}/F^+((\zeta_N))), \text{ad}^0 r(1) \otimes A/T^m) \to H^1(\text{Gal}(L_{\infty}/F_{\infty}^+), \text{ad}^0 r(1) \otimes A/T^m).$$

The first (resp. third) terms in the sequence are bounded independently of $N$ and $m$ by the second (resp. third) part of Lemma 5.3 above. Thus the same is true for the middle term. We now have an inclusion

$$0 \to H^1(\text{Gal}(L_m,N/F^+((\zeta_N))), \text{ad}^0 r(1) \otimes A/T^m) \to H^1(\text{Gal}(L_{\infty}/F^+((\zeta_N))), \text{ad}^0 r(1) \otimes A/T^m),$$

giving the lemma. \qed
Lemma 5.6. We can find a constant $C > 0$, an integer $q \geq n(n-1)/2[F^+:\mathbb{Q}]$ and for each $N \geq 1$ a choice of Taylor-Wiles data $(Q_N, Q_N, \{\tau_v\}_v \in Q_N)$ of level $N$ and order $q$ such that:

(i) For all $N, m \geq 1$,

$$H^1_{S^+,T}(G_{F^+,S_N}, \text{ad} \, r(1) \otimes A/T^m)$$

is a finite $A$-module of cardinality bounded by $C$.

(ii) For each $N \geq 1$, there is an isomorphism of $A$-modules

$$H^1_{S^+,T}(G_{F^+,S_N}, \text{ad} \, r \otimes A/E/A) \cong (E/A)^{q-n(n-1)/2[F^+:\mathbb{Q}]} \oplus T(N),$$

where $T(N)$ is a finite $A$-module of cardinality bounded by $C$.

Proof. By Proposition 5.4, the second part of the lemma will follow from the first. Suppose given a tuple $(Q_N, Q_N, \{\tau_v\}_v \in Q_N)$ of Taylor-Wiles data of level $N$, and consider adding an extra place $u$ to $Q_N$ to obtain $Q'_N = Q_N \cup \{u\}$, $Q''_N = Q_N \cup \{\tilde{u}\}$ for some place $\tilde{u}$ of $F$ above $u$, and choosing an eigenvalue $\tau_u$ of $\tau(\text{Frob}_{\tilde{u}})$ to obtain a new choice of Taylor-Wiles data:

$$(Q'_N, \tilde{Q}'_N, \{\tau_v\}_v \in Q_N \cup \{\tau_u\}).$$

Then, writing $S_N$ and $S'_N$ for the respective augmented deformation problems, we have for each $m \geq 1$ a commutative diagram with exact rows and Cartesian squares

$$
\begin{array}{c}
0 \longrightarrow H^1_{S^+,T}(G_{F^+,S'_N}, \text{ad} \, r(1) \otimes A/T^m) \longrightarrow H^1_{S^+,T}(G_{F^+,S_N}, \text{ad} \, r(1) \otimes A/T^m) \longrightarrow A/T^m \\
0 \longrightarrow H^1_{S^+,T}(G_{F^+,S'_N}, \text{ad} \, r(1) \otimes A/E/A) \longrightarrow H^1_{S^+,T}(G_{F^+,S_N}, \text{ad} \, r(1) \otimes A/E/A) \longrightarrow E/A
\end{array}
$$

the last arrow in each row given on cocycles by the map $\phi \mapsto \text{tr} \, e_{\text{Frob}_{\tilde{u}}, \tau_u} \phi(\text{Frob}_{\tilde{u}})$, where $e_{\text{Frob}_{\tilde{u}}, \tau_u}$ is by definition the unique $r(\text{Frob}_{\tilde{u}})$-equivariant projection of $A^n$ onto a direct summand $A$-module lifting the $\tau_u$-eigenspace of $\tau(\text{Frob}_{\tilde{u}})$.

Suppose that $H^1_{S^+,T}(G_{F^+,S_N}, \text{ad} \, r(1) \otimes A/E/A) \cong (E/A)^r \oplus X$, where $X$ is a finite $A$-module, annihilated by $T^M$. Suppose that there exist integers $K, m > 0$ and a cocycle $\phi \in H^1_{S^+,T}(G_{F^+,S_N}, \text{ad} \, r(1) \otimes A/T^m)$ such that the cohomology class of $\phi$ has exact order $m$ and the image of this class in $A/T^m$ has exact order at least $m - K$, with $m - K > M$. Then there is an isomorphism $H^1_{S^+,T}(G_{F^+,S_N}, \text{ad} \, r(1) \otimes A/E/A) \cong (E/A)^{r-1} \oplus X'$, where $|X'| \leq |X| \times |A/T^K|$. By induction, to prove the lemma it therefore suffices to show that for any $Q_N$ and cocycle $\phi$ as above, we can find an extra place $u$ and integers $K, c$ satisfying this requirement, with $K$ independent of the choice of $Q_N$.

Using the decomposition $\text{ad} \, r(1) = \text{ad}^0 \, r(1) \oplus \lambda(1)$, we can suppose that $\phi$ is valued either in $\text{ad}^0 \, r(1) \otimes A/T^m$ or $\lambda(1) \otimes A/T^m$. Suppose first that $\phi$ is valued in $\text{ad}^0 \, r(1) \otimes A/T^m$, and that $[\phi]$ has exact order $m$. By Lemma 5.3, the image of $[\phi]$ in $H^1(G_{L_m,N}, \text{ad}^0 \, r(1) \otimes A/T^m)^{G_{F^+}}$ has order at least $m - K_0 - K_1$. Let us write $f$ for this restriction, which can be viewed as a homomorphism $f : G_{L_m,N} \rightarrow \text{ad}^0 \, r(1) \otimes A/T^m$ whose image is invariant under the action of $G_{F^+}$. Moreover, the image of $f$ contains an element of exact order $m - K_0 - K_1$.

By the third part of Lemma 5.3, we have

$$\text{image} \, f \supset T^{K_0 + K_1 + K_2} \, \text{ad}^0 \, r(1) \otimes A/T^m.$$ 

Choose $\sigma \in G_{F^+}$ such that $\tau(\sigma)$ has an eigenvalue $\overline{\sigma}$ of multiplicity $1 \leq p \leq n - 1$, with $(p, l) = 1$, and such that $\tau(\sigma)$ acts semisimply on its $\overline{\sigma}$-generalized eigenspace. (It is easy to see that such a $\sigma$ exists.)
If \( \text{tr} e_\sigma, \pi \phi(\sigma) \) has exact order at least \( m - K_0 - K_1 - K_2 \), then let \( \sigma_0 = \sigma \). Otherwise, we can find \( \tau \in G_{L_{m,N}} \) such that \( \text{tr} e_\sigma, \tau f(\tau) \) has exact order at least \( m - K_0 - K_1 - K_2 \). Now set \( \sigma_0 = \tau \sigma \). In this case we have \( \phi(\sigma_0) = \phi(\tau) + \tau \phi(\sigma) \), and \( \phi(\sigma_0) \) also has order at least at least \( m - K_0 - K_1 - K_2 \).

In either case, we see that \( \text{tr} e_\sigma, \pi \phi(\sigma_0) \) has exact order at least \( m - K_0 - K_1 - K_2 \). By the Chebotarev density theorem, we can now find a place \( u \) of \( F^+ \) with extension \( \overline{u} \) to \( F \), split in \( F^+(\zeta_N) \), such that \( \text{tr} e_{\text{Frob}_u}, \pi \phi(\text{Frob}_u) \) has exact order at least \( m - K_0 - K_1 - K_2 \). Taking \( m > M + K_0 + K_1 + K_2 \) completes the induction step in this case.

Now suppose instead that \( \phi \) is valued in \( \mathfrak{z}(1) \). The restriction of \( \phi \) to an element of the group

\[
H^1(G_{F(\zeta_N)}, \mathfrak{z} \otimes_A A/T^m)
\]

has exact order at least \( m - K_0 \). We can view this restriction as a homomorphism

\[
\phi : G_{F(\zeta_N)} \rightarrow \mathfrak{z} \otimes_A A/T^m.
\]

Choose \( \sigma \in G_{F(\zeta_N)} \) such that \( \phi(\sigma) \) has exact order at least \( m - K_0 \). Now \( \ker \phi \) projects surjectively onto \( \tau(G_{F(\zeta_N)}) \), since this latter group has no non-trivial quotients of \( l \)-power order. In particular, we can choose \( \tau \in \ker \phi \) such that \( \tau | \sigma(\sigma) \) has an eigenvalue \( \alpha \) of multiplicity \( p \) prime to \( l \), and acts semisimply on its \( \sigma \)-generalized eigenspace. Taking \( \sigma_0 = \tau \sigma \), we have \( \text{tr} e_{\sigma_0, \pi} \phi(\sigma_0) = p \phi(\sigma) \), which is therefore of exact order \( m - K_0 \). Applying the Chebotarev density theorem once more completes the induction step in this case, and thereby the proof of the lemma.

**Corollary 5.7.** Suppose that we are in the situation of \([4.6]\) and that the hypotheses of this section hold for \( r = r_\pi \) and \( S = S_\chi \). Then hypothesis (i) of Theorem \([4.17]\) holds.

**Proof.** With notation as above, we must construct an isomorphism of \( A \)-modules

\[
\text{Hom}_A(\overline{p}_N/(\overline{p}_N \text{loc} + \overline{p}_N), E/A) \cong H^1_{S_{N,T}}(G_{F^+, S_N}, \text{ad} r \otimes_A E/A).
\]

The first term here is naturally isomorphic to

\[
\text{Hom}_A(\overline{R}_{S_{N,T}/\overline{p}_N}^{\text{loc}}, A \oplus \epsilon E/A).
\]

Let \( (r; \alpha_v)_{v \in T} \) denote a representative of the \( T \)-framed deformation of \( \tau \) over \( A \) corresponding to the ideal \( \overline{p}_N \). The above group is in bijection with the set of equivalence classes of \( T \)-framed liftings \( (\overline{\tau}; \overline{\alpha}_v)_{v \in T} \) of \( \tau \) to \( A \oplus \epsilon E/A \) which are equivalent to \((r; \alpha_v)_{v \in T} \) after reduction modulo \( \epsilon \), are of type \( S_N \), and such that for each \( v \in T \), \( \overline{\tau}_v | G_{F_v} \overline{\alpha}_v \) is equal to \( r_{1_{G_{F_v}}} | G_{F_v} \overline{\alpha}_v \). This set is itself in bijection with the set of \( T \)-framed liftings \( (\overline{\tau}; \overline{\alpha}_v)_{v \in T} \) over \( A \oplus \epsilon E/A \) which are equal to \((r; \alpha_v)_{v \in T} \) after reduction modulo \( \epsilon \), are of type \( S_N \), and such that for all \( v \in T \), \( \overline{\tau}_v | G_{F_v} \overline{\alpha}_v = r_{1_{G_{F_v}}} | G_{F_v} \overline{\alpha}_v \), taken up to \( 1 + \epsilon M_n(E/A) \)-conjugation. (It is an abuse of language to speak of liftings to \( A \oplus \epsilon E/A \), since this ring does not lie in \( C_H \); however, this does not cause problems, cf. the discussion before \([\text{CHT}08 \ \text{Definition 2.2.2}]\).

Given such a \( T \)-framed lifting \( (\overline{\tau}; \overline{\alpha}_v)_{v \in T} \), we write \( \overline{\tau} = (1 + \epsilon \phi) \overline{r} \), with \( \phi \in Z^1(G_{F^+, S_N}, \text{ad} r \otimes_A E/A) \), and \( \overline{\alpha}_v = \alpha_v + \epsilon \psi_v \), \( \psi_v \in M_n(E/A) \). The cohomology class

\[
[(\phi, (\psi_v)_{v \in T})] \in H^1_{S_{N,T}}(G_{F^+, S_N}, \text{ad} r \otimes_A E/A)
\]

then depends only on \( (\overline{\tau}; \overline{\alpha}_v)_{v \in T} \) up to \( 1 + \epsilon M_n(E/A) \)-conjugation, and it is now easy to check (following \([\text{CHT}08 \ \text{Proposition 2.2.9}]\)) that this assignment gives the desired isomorphism of \( A \)-modules. \( \square \)

## 6 The main argument

In this section we combine the results of the previous two sections to prove the analogue of an \( R = \mathbb{T} \) theorem in our context. We take up the notations of the beginning of \([4.3]\). Thus \( L \) is an imaginary CM field with totally real subfield \( L^+ \), \( G \) is a unitary group over \( L^+ \) of dimension \( n \), and \( S = T = S_l \cup R \cup S(B) \cup S_n \) is a set of primes of \( L^+ \) split in \( L \). We fix an open compact subgroup \( U = \prod_u U_v \) of \( G(\mathbb{A}_F^\infty) \) having the following form:

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For \( v \) inert in \( L \), \( U_v \subset G(L_v^+) \) is a hyperspecial maximal compact subgroup.

For \( v \notin T \) split in \( L \), \( U_v = G(O_{L_v^+}) \).

For \( v \in S(B), U_v \) is the unique maximal compact subgroup.

For \( v \in S_t, U_v = G(O_{L_v^+}) \).

For \( v \in S_a, U_v = \ell_v^{-1} \ker(GL_n(O_{L_v}) \to GL_n(k(\bar{v}))) \).

For \( v \in R, U_v = \ell_v^{-1} \text{Iw}(\bar{v}) \).

We suppose that \( m \subset \mathbb{T}_1(U(t^\infty), O) \) is a residually Schur maximal ideal, giving rise to a residual Galois representation \( \tau_m : G_{L+s} \to G_n(k) \). We suppose that \( \tau_m \) satisfies the following hypotheses:

(i) The quotient \( R_{S_1}^{\text{red}} \) of \( R_{S_1}^{\text{univ}} \) classifying reducible deformations is finite over \( \Lambda \) and of dimension bounded above by \( n[L^+ : \mathbb{Q}] - r(n(1) - 5) \).

(ii) The prime \( l \) is strictly greater than 3, and for each \( v \in R \), the highest power of \( l \) dividing \( q_v - 1 \) is strictly greater than \( n \).

(iii) For each \( v \in S_t \), we have \( [L_v : \mathbb{Q}] > \sup(r(n(1) + 5), n(n - 1)/2 + 1) \), where \( r = |R| \).

Let \( r : G_{L+s} \to G_n(O) \) be a lifting of \( \tau_m \) of type \( S_1 \) such that \( r|_{G_L} \) is ordinary of weight \( \lambda \), for some \( \lambda \in (\mathbb{Z}_+^n)_{\text{Hom}(L, \mathbb{Q})} \). Then \( r \) is automorphic of weight \( \lambda \).

We suppose that \( S_a \) is non-empty and that for every \( v \in S_a, v \) is absolutely unramified, \( \tau_m \) is unramified above \( v \), \( \text{ad} \tau(\text{Frob}_v) = 1 \) and \( v \) does not split in \( L(\zeta) \). Then \( H^1(G_{L_v}, \text{ad} \tau(1)) = 0 \), and \( U \) is sufficiently small. Suppose also that for each \( v \in S_t \cup R \cup S(B), \tau_m|_{G_{L_v}} \) is trivial and \( q_v \equiv 1 \mod l \). Under these assumptions we have defined a global deformation problem

\[
S_1 = \left( \frac{L}{L^+}, T, \mathbb{T}, \Lambda, \tau_m, e^{1-n} \varpi_{L/L^+}, \{R_v^{\Delta}\}_{v \in S_t} \cup \{R_v^{\text{St}}\}_{v \in S(B)} \cup \{R_v^{\text{Iw}}\}_{v \in S_a} \cup \{R_v^{\text{Iw}}\}_{v \in R} \right).
\]

This section is devoted to the proof of the following theorem.

**Theorem 6.1.** With assumptions as above, suppose further that:

(i) The quotient \( R_{S_1}^{\text{red}} \) of \( R_{S_1}^{\text{univ}} \) classifying reducible deformations is finite over \( \Lambda \) and of dimension bounded above by \( n[L^+ : \mathbb{Q}] - r(n(1) - 5) \).

(ii) The prime \( l \) is strictly greater than 3, and for each \( v \in R \), the highest power of \( l \) dividing \( q_v - 1 \) is strictly greater than \( n \).

(iii) For each \( v \in S_t \), we have \( [L_v : \mathbb{Q}] > \sup(r(n(1) + 5), n(n - 1)/2 + 1) \), where \( r = |R| \).

Let \( r : G_{L+s} \to G_n(O) \) be a lifting of \( \tau_m \) of type \( S_1 \) such that \( r|_{G_L} \) is ordinary of weight \( \lambda \), for some \( \lambda \in (\mathbb{Z}_+^n)_{\text{Hom}(L, \mathbb{Q})} \). Then \( r \) is automorphic of weight \( \lambda \).

Let us say that a soluble CM extension \( M/L \) is good if it is linearly disjoint from the extension of \( L(\zeta) \) cut out by \( \tau_m|_{G_{L(\zeta)}} \) and every prime above \( S_t \cup S_a \cup R \) splits in \( M \). If \( M/L \) is a good extension we have constructed in \( \S 4.3 \) a deformation problem \( S_{1,M} \) and a diagram of \( \Lambda_M \)-algebras

\[
\begin{array}{ccc}
R_{S_1}^{\text{univ}} & \longrightarrow & P_{S_1} \\
\uparrow & & \uparrow \\
R_{S_{1,M}}^{\text{univ}} & \longrightarrow & P_{S_{1,M}} \\
\downarrow & & \downarrow \\
T_1^T(U(t^\infty), O)_m & \to & T_1^T(U(t^\infty), O)_n_M \\
\end{array}
\]

We have defined an ideal

\[
J_{S_{1,M}} = \ker \left( P_{S_{1,M}} \to T_1^T(U(t^\infty), O)_n_M \right).
\]
We write \( J_M = J_{S_1,M} P_{S_1} \). Let \( p \subset R_{S_1}^{\text{univ}} \) be a prime ideal. We say that \( p \) is potentially pro-automorphic if there exists a good extension \( M/L \) such that \( J_M \subset p \).

Let \( p \subset R_{S_1}^{\text{univ}} \) be a prime ideal of dimension one and characteristic \( l \). For each \( v \in S_1 \) there are universal characters \( \psi_1^v, \ldots, \psi_n^v : I_{L_v} \to A^\times \). Let \( A \) denote the normalization of \( R_{S_1}^{\text{univ}}/p \), and \( E = \text{Frac} A \). We say that \( p \) is generic if it satisfies the following properties:

- The representation \( r_p|_{G_L} \otimes_A E \) is absolutely irreducible.
- For each \( v \in S_1 \), the characters \( \psi_1^v, \ldots, \psi_n^v \) are distinct modulo \( p \).
- There exists \( v \in S_1 \) and \( \sigma \in I_{L_v} \) such that the elements \( \psi_1^v(\sigma) \mod p, \ldots, \psi_n^v(\sigma) \mod p \in A^\times \) satisfy no non-trivial \( \mathbb{Z} \)-linear relation.

The interest of these concepts is the following consequence of our work so far.

**Proposition 6.2.** Let \( p \subset R_{S_1}^{\text{univ}} \) be a prime which is potentially pro-automorphic and generic. Suppose further that for each \( v \in R \), the restriction \( r_p|_{G_{L_v}} \) is trivial. Then every minimal prime \( Q \subset p \) is potentially pro-automorphic.

**Proof.** By hypothesis, there exists a good extension \( M_0/L \) such that \( J_{M_0} \subset p \). By making a further soluble extension, we can find a good extension \( M_0/L \) containing \( M_0 \) such that for every prime \( \bar{w} \) of \( M_1 \) above a prime of \( S(B) \), \( r_p|_{G_{M_1,\bar{w}}} \) is unramified and \( r_p(\text{Frob}_{\bar{w}}) \) is scalar. In fact, if \( t_\bar{w} \) denotes a generator of the \( l \)-part of tame inertia at the place \( \bar{w} \), \( r_p(t_\bar{w}) \) is a unipotent matrix in \( GL_n(A) \), hence of finite (and \( l \)-power) order. After making a finite local extension to kill off the image of inertia, Frobenius is mapped to a unipotent element times a scalar matrix (since \( q_v = 1 \mod l \)). A further \( l \)-power extension now gives a local representation of the desired form.

Then \( J_{M_1} \subset J_{M_0} \), by Proposition 4.16. Let \( p_{M_1} \subset R_{S_1,M_1}^{\text{univ}} \) denote the pullback of \( p \). Then \( J_{S_1,M_1} \subset p_{M_1} \). Let \( Q \) be as in the proposition, and let \( Q_{M_1} \) denote its pullback to \( R_{S_1,M_1}^{\text{univ}} \). We will show that \( J_{S_1,M_1} \subset Q_{M_1} \). This will imply \( J_{M_1} \subset Q \), which is what we need to prove. By Lemma 3.26 and Proposition 5.1 \( r_p|_{G_{M_1}} \) is absolutely irreducible and so \( p_{M_1} \) is generic. Arguing as in the proof of Lemma 3.34, we can find a character \( \psi : G_{M_1,\bar{w}} \to 1 + m_A \) such that \( p_{M_1,\bar{w}} \) is defined and satisfies hypotheses (ii)–(v) of Theorem 4.17. In particular, we have \( J_{S_1,M_1} \subset p_{M_1,\psi} \) by Corollary 4.12. By Lemma 5.6 it also satisfies hypothesis (i). Let \( Q' \subset Q_{M_1} \) be a minimal prime of \( R_{S_1,M_1}^{\text{univ}} \). Then \( Q' \subset p_{M_1} \), so \( Q' \subset p_{M_1,\psi} \), by Lemma 3.34 Corollary 4.18, now implies that \( J_{S_1,M_1} \subset Q' \subset Q_{M_1} \). This completes the proof of the proposition.

To move between generic primes, we use the notion of connectedness dimension defined in §3.8.

**Lemma 6.3.** We have \( c(R_{S_1}^{\text{univ}}) \geq n[L^+ : \mathbb{Q}] - rn - 2 \).

**Proof.** We first give a lower bound for \( c(R_{S_1,T}^{\text{loc}}) \). Recall that by definition we have

\[
R_{S_1,T}^{\text{loc}} = \left( \bigotimes_{v \in S_a} R_v^{\text{loc}} \right) \otimes \left( \bigotimes_{v \in S(B)} R_v^{\text{St}} \right) \otimes \left( \bigotimes_{v \in S_1} R_v^{\Delta} \right) \otimes \left( \bigotimes_{v \in R} R_v^{\text{univ}} \right).
\]

By Lemma 3.3, the minimal primes of the ring

\[
R_0 = \left( \bigotimes_{v \in S_a} R_v^{\text{loc}} \right) \otimes \left( \bigotimes_{v \in S(B)} R_v^{\text{St}} \right) \otimes \left( \bigotimes_{v \in S_1} R_v^{\Delta} \right)
\]

are in bijection with the minimal primes \( Q \subset \Lambda \). In particular, for any two minimal primes \( Q_1, Q_2 \subset R_0 \), \( \dim R_0/(Q_1 + Q_2) \geq \dim R_0/\lambda = \dim R_{S_1,T}^{\text{loc}} - 1 - rn^2 \). By definition of the connectedness dimension, \( c(R_0) \geq \dim R_{S_1,T}^{\text{loc}} - 1 - rn^2 \). It follows from the description of \( R_1^{\text{loc}} \) in §3 that this ring admits a presentation as a quotient of a power series ring over \( O \) in \( 2n^2 \) variables by \( n^2 + n \) relations. Similarly, \( R_{S_1,T}^{\text{loc}} \) admits a presentation as a quotient of a power series ring over \( R_0 \) in \( 2rn^2 \) variables by \( rn(n+1) \) relations. Using Proposition 3.35 we see that

\[
c(R_{S_1,T}^{\text{loc}}) \geq \dim R_{S_1,T}^{\text{loc}} - 1 - 2rn^2 + 2rn^2 - rn(n+1) - 1 = [L^+ : \mathbb{Q}]n(n+1)/2 + n^2|T| - rn - 1.
\]
Applying Proposition 3.35 once more to the presentation of $R_{S_1}^{\text{red}}$ given in Proposition 3.9, we find
\[ c(R_{S_1}^{\text{red}}) \geq c(R_{S_1}^{\text{red}, r}) - n(n - 1)/2[L^+ : \mathbb{Q}] - 1 \geq n[L^+ : \mathbb{Q}] + n^2|T| - rn - 2. \]

Finally, it is clear from the definition of connectedness dimension that $c(R_{S_1}^{\text{red}}) = c(R_{S_1}^{\text{uni}}) + n^2|T|$.

\[ \square \]

**Lemma 6.4.** Let $R$ be an object of $\mathcal{C}_k$ of dimension $d \geq 1$, and suppose given countably many ideals $I_1, I_2, \ldots$ such that for all $i$, we have $\dim R/I_i \leq d - 1$. Then there exists a dimension one prime $p \subset R$ such that $p$ does not contain $I_i$ for any $i$.

**Proof.** If $d = 1$ then the result is clear. Otherwise, by the Noether normalization theorem for complete local rings, we can find an injective finite map $k[[x_1, \ldots, x_d]] \hookrightarrow R$. We may therefore assume $R = k[[x_1, \ldots, x_d]]$ and that each $I_i = (f_i)$ is principal. Then $R$ is a UFD and there exist uncountably many pairwise non-associate prime elements $g \in \mathfrak{m}_R - \mathfrak{m}_R^2$, as follows easily from the Weierstrass preparation theorem. Choosing $g$ coprime to each $f_i$ and passing to $R/(g)$, we can reduce by induction to the case $d = 1$. \[ \square \]

**Proof of Theorem 6.1.** Let $R^{\text{red}} \subset R_{S_1}^{\text{uni}}$ denote the ideal cutting out the subspace of reducible deformations, and let $R^{\text{red}} \subset \Lambda$ denote the pullback of $R^{\text{red}}$ to $\Lambda$. Our hypotheses imply that the dimension of $\Lambda/R^{\text{red}}$ is at most $n[L^+ : \mathbb{Q}] - mn(n + 1) - 5$.

For each $v \in I$, let $\sigma_v^1, \ldots, \sigma_v^n$ denote a basis of a maximal free $\mathbb{Z}_l$-summand of $\mathcal{O}_{L_v}(l)$, where $d_v = [L_v : \mathbb{Q}_l]$. For each $i, j, v \in S_1$, define an ideal
\[ I(i, j, v) = (\lambda, \{\psi_i^v(\sigma_j) - \psi_j^v(\sigma_k)\}_{k=1}^{v=\#d_v}) \subset \Lambda. \]

Then $\Lambda/I(i, j, v)$ has dimension $n[L^+ : \mathbb{Q}] - n d_v$. On the other hand, suppose given for each $v \in S_1$ an $n \times d_v$ matrix of integers $a_{i,j}^v$ such that each column contains a non-zero entry. Let $J(a_{i,j}^v)$ denote the ideal of $\Lambda$ generated by $\lambda$ and the elements
\[ \left( \prod_{i=1}^n \psi_i^v(\sigma_j)^{a_{i,j}} \right) - 1 \text{ as } j = 1, \ldots, d_v \text{ and } v \in S_1. \]

Then $\Lambda/J(a_{i,j}^v)$ has dimension $(n - 1)[L^+ : \mathbb{Q}]$.

Together $R^{\text{red}}, I(i, j, v)$ and $J(a_{i,j}^v)$ define a countable collection of ideals of $\Lambda$ whose quotients have dimension bounded above by $n[L^+ : \mathbb{Q}] - mn(n + 1) - 5$. It follows from the previous lemma that for any good extension $M/L$, any quotient of $R_{S_1}^{\text{uni}}/(\lambda, J_M)$ of dimension at least $n[L^+ : \mathbb{Q}] - mn(n + 1) - 4$ contains a generic potentially pro-automorphic prime $p$. (Note that $R_{S_1}^{\text{uni}}/J_M$ is finite over $\Lambda$, since it is finite over $R_{S_1, M}^{\text{uni}}/J_{S_1, M}$, hence over $P_{S_1, M}/J_{S_1, M} = \prod_{i=1}^l (U_M(l^{\infty}), \mathcal{O})_{m,M}$, hence over $\Lambda_M$.)

Fix a choice of lifting $t_{S_1}^{\text{uni}}$ representing the universal deformation. This induces for each $v \in R$ a homomorphism $R_i^\mathfrak{l} \to R_{S_1}^{\text{uni}}$, and we let $J_R$ denote the ideal generated by the images of $\mathfrak{m}_R$, $v \in R$. This ideal is independent of the choice of lifting, and for any quotient $R_{S_1}^{\text{uni}}/I$ of characteristic $l$, we have $\dim R_{S_1}^{\text{uni}}/(J_R, I) \geq \dim R_{S_1}^{\text{uni}}/I - rn^2$. It follows that there exists a generic prime $p \subset R_{S_1}^{\text{uni}}/(J_R, J_L)$, since $\dim R_{S_1}^{\text{uni}}/J_L \geq \dim \Lambda = 1 + n[L^+ : \mathbb{Q}]$. By Proposition 6.2, any minimal prime $Q \subset p$ of $R_{S_1}^{\text{uni}}$ is potentially pro-automorphic.

We now consider the partition of the set of minimal primes of $R_{S_1}^{\text{uni}}$ into two sets $C_1, C_2$, consisting of those primes which respectively are and are not potentially pro-automorphic. We have shown that $C_1$ is non-empty. We claim that $C_2$ is empty. For otherwise, there exist minimal primes $Q_1 \in C_1, Q_2 \in C_2$ such that
\[ \dim R_{S_1}^{\text{uni}}/(Q_1, Q_2) \geq c(R_{S_1}^{\text{uni}}) \geq n[L^+ : \mathbb{Q}] - rn - 2, \]

and hence
\[ \dim R_{S_1}^{\text{uni}}/(Q_1, Q_2, J_R) \geq n[L^+ : \mathbb{Q}] - rn - rn^2 - 3 = n[L^+ : \mathbb{Q}] - rn(n + 1) - 3. \]

In particular, this ring contains a generic potentially pro-automorphic prime $p$. Applying Proposition 6.2 once more, we deduce that $Q_2$ is potentially pro-automorphic, a contradiction.
Now let \( r : G_{L^+} \to \mathcal{G}_A(\mathcal{O}) \) be a lifting of \( \rho_m \) which is ordinary of weight \( \lambda \) and of type \( S_1 \), as in the statement of the theorem. This induces a homomorphism \( R^n_{S_n^\text{uni}} \to \mathcal{O} \). Let \( Q \) be a minimal prime contained inside the kernel of this homomorphism. Then there is a good extension \( M/L \) such that \( J_M \subset Q \), and so the induced homomorphism \( R^n_{S_n^\text{uni}} \to \mathcal{O} \) kills \( J_{S_n,M} \), and the map \( P_{S_n,M} \to \mathcal{O} \) induced by \( r \) factors through \( \mathbb{T}_1^{T_M}(U_M(\mathcal{O}^\infty), \mathcal{O})_{m,M} \). It now follows from \( \text{[Ger Lemma 2.6.4]} \) and \( \text{[CHT08 Proposition 3.3.2]} \) that \( r|_{G_M} \) is automorphic. We now apply \( \text{[BLGHT11 Lemma 1.4]} \) to deduce that \( r|_{G_L} \) is automorphic of weight \( \lambda \). (Note that this result requires that \( r|_{G_M} \) be irreducible, which is implied by \( \text{[TY07 Corollary B]} \).)

\[ \square \]

7 The main theorem

Let \( l > 3 \) be a prime. Let \( K \) be a finite extension of \( \mathbb{Q}_l \) inside \( \mathbb{Q}_l \), with ring of integers \( \mathcal{O} \) and residue field \( k \). In this section we prove the following result.

**Theorem 7.1.** Let \( F/F^+ \) be a CM extension of a totally real field, and let \( n \geq 2 \) be an integer. Suppose that \( \rho : G_F \to \text{GL}_n(K) \) is a continuous semisimple representation satisfying the following hypotheses:

\[
\begin{align*}
(i) \quad \rho' &\cong \rho^n \epsilon^{1-n}.
(ii) \quad \rho \text{ is ramified at only finitely many places. }
(iii) \quad \rho \text{ is ordinary of weight } \lambda, \text{ for some } \lambda \in \mathcal{Z}'_k^\text{Hom}(F,\mathcal{O}_l).
(iv) \quad F(\mathbb{Q}_l) \text{ is not contained in } F^{\text{ner ad}}(\mathfrak{F}').
(v) \quad \mathfrak{p}^n_s \cong \mathfrak{p}_1 \oplus \mathfrak{p}_2, \text{ where the } \mathfrak{p}_i|_{G_F(\mathbb{Q}_l)} \text{ are adequate, in the sense of \text{[Tho12 §2]}, and primitive.}
(vi) \quad \mathfrak{p}_1 \neq \mathfrak{p}_2 \text{ and } \epsilon^{1-n}\mathfrak{p}_1 \neq \mathfrak{p}_2.
(vii) \quad \text{There exists a finite place } \mathfrak{v}_0 \text{ of } F, \text{ coprime to } l, \text{ such that } \rho|_{G_{F_{\mathfrak{v}_0}}} \cong \bigoplus_{i=1}^n \psi \epsilon^{n-i} \text{ for some unramified character } \psi : G_{F_{\mathfrak{v}_0}} \to K^\times.
(viii) \quad \text{There exists a RACSDC representation } \pi \text{ of } \text{GL}_n(k_f) \text{ and } \iota : \mathbb{Q}_l \cong \mathbb{C} \text{ such that:}
\begin{align*}
(a) \quad & \pi \text{ is } \iota\text{-ordinary. }
(b) \quad \overline{r_1(\pi)}^{n_s} \cong \mathfrak{p}^n_s.
(c) \quad \pi_{\mathfrak{v}_0} \text{ is an unramified twist of the Steinberg representation. }
\end{align*}
(ix) \quad \text{There exists a CM extension } F_0/F \text{ linearly disjoint from the extension of } F(\mathbb{Q}_l) \text{ cut out by } \mathfrak{p}^n_s|_{G_F(\mathbb{Q}_l)} \text{ and } \text{RAECSDC representations } (\pi_1, \chi_1), (\pi_2, \chi_2) \text{ of } \text{GL}_n(k_{F_0}) \text{ and } \text{GL}_{n_2}(k_{F_0}), \text{ respectively, such that } \pi_1, \pi_2 \text{ are } \iota\text{-ordinary and } \overline{r_i(\pi_i)} \cong \mathfrak{p}_i|_{G_{F_0}} \text{ for } i = 1, 2.
\end{align*}
\]

Then \( \rho \) is automorphic.

We note that hypothesis (ix) is often satisfied in practice, by potential automorphy theorems (cf. \text{[BLGHT16]}).

**Proof.** We may suppose without loss of generality that \( v_0 \) is split over \( F^+ \). After possibly enlarging the field \( K \), we can find a self-dual \( \mathcal{O} \)-lattice for \( \rho \) inside \( K \), so we can view \( \rho \) as a representation \( G_F \to \text{GL}_n(\mathcal{O}) \) such that \( \mathfrak{p} : G_F \to \text{GL}_n(k) \) is conjugate-self-dual (see \text{[CHT08 Lemma 2.1.5]}). We do the same for \( r_1(\pi) \) to obtain a homomorphism \( \rho' : G_F \to \text{GL}_n(\mathcal{O}) \). Our hypotheses imply that \( \mathfrak{p} \) is semisimple, so we may suppose that \( \mathfrak{p} = \mathfrak{p}' \).

We claim that we can find a soluble CM extension \( L \) of \( F \) with totally real subfield \( L^+ \) such that:
(i) Every prime above \( l \) or a prime at which \( \rho \) or \( \pi \) is ramified splits in \( L/L^+ \), and \( L/L^+ \) is unramified at all finite places.

(ii) \( L \) is linearly disjoint over \( F \) from the extension of \( F_0(\zeta_l) \) cut out by \( \overline{\mathcal{P}}|_{G_{F_0(\zeta_l)}} \).

(iii) The number of places of \( L \) above \( \overline{v}_0 \) is even, and \( |L^+:\mathbb{Q}| \) is divisible by 4.

(iv) There exists a prime \( \overline{v}_1 \) of \( L \) split over \( L^+ \) and absolutely unramified such that \( \rho \) and \( \pi \) are unramified at the prime of \( F \) below \( \overline{v}_1 \), \( \text{ad}(\text{Frob}_{\overline{v}_1}) \) is trivial, and \( \overline{v}_1 \) does not split in \( L(\zeta_l) \).

(v) For each place \( v \) of \( L \) above a place at which \( \rho \) or \( \pi \) is ramified, or dividing \( l \), \( \overline{\mathcal{P}}|_{G_{L,v}} \) is trivial and \( q_v = 1 \mod l \).

(vi) Let \( R \) denote the set of places \( v \) of \( L^+ \) not dividing \( \overline{v}_0 \) or \( l \) above which \( \rho|_{G_L} \) or \( \pi_L \) is ramified, and let \( r = |R| \). Then for each prime \( v \) of \( L \), \( |L_v : \mathbb{Q}_l| > \sup \{rn(n+1), n(n-1)/2 + 1 \} \). For each \( v \in R \), the highest power of \( l \) dividing \( q_v - 1 \) is strictly greater than \( n \).

(vii) Let \( \Delta \) denote the Galois group of the maximal abelian \( l \)-extension of \( L \) unramified outside \( l \), and let \( \Delta_0 \) denote the Galois group of the maximal abelian \( l \)-extension of \( L \) unramified outside \( l \) in which every prime dividing \( v_0 \) splits completely, where \( v_0 \) is the place of \( F^+ \) below \( \overline{v}_0 \). Let \( c \in \text{Gal}(L/L^+) \) be complex conjugation. Then we have

\[
\dim_{\mathbb{Q}_l} \ker(\Delta \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \to \Delta_0 \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)^{c=-1} > 6 + rn(n+1).
\]

To see this we proceed as follows. Choose a soluble CM extension \( M \) of \( F \) satisfying the first 5 items. Note that for any odd integer \( d \geq 1 \), we can choose a cyclic totally real extension \( M' \) of \( F^+ \) of degree \( d \) in which \( v_0 \) splits and in which every place dividing \( l \) or a place at which \( \pi \) or \( \rho \) is ramified is totally inert, and such that \( L = M \cdot M' \) still satisfies the first 5 items above, by [AT09, §X.2, Theorem 5]. We claim that choosing \( d \) to be very large and coprime to the absolute residue degrees of the primes \( v \) of \( M^+ \) in \( R \) or dividing \( l \), \( L \) will also satisfy the remaining bullet points. Indeed, only the last point remains to be checked.

For this we recall that

\[
\dim_{\mathbb{Q}_l} \ker(\Delta \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \to \Delta_0 \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)^{c=-1} = \text{rank}_{\mathbb{Z}_l} \left( \mathcal{O}_{L,S_0}^\times \right)^{c=-1}
\]

where \( S_0 \) denotes the set of primes of \( L \) above \( v_0 \). Here the overline denotes the closure of the image of the units inside \( \prod_{v \in S_0} \mathcal{O}_{L,v}^\times(l) \). Let \( w_0 \) denote a place of \( M^+ \) above \( v_0 \), and let \( S_1 \) denote the set of places of \( L \) above \( w_0 \). Then

\[
\text{rank}_{\mathbb{Z}_l} \left( \mathcal{O}_{L,S_0}^\times \right)^{c=-1} \geq \text{rank}_{\mathbb{Z}_l} \left( \mathcal{O}_{L,S_1}^\times \right)^{c=-1},
\]

and the extension \( L/M^+ \) is abelian. [Jan85, Théorème 3] now implies that the latter quantity is equal to \( d \), since \( w_0 \) splits in \( L \), by construction. (We note that in [Jan85] this theorem is stated only for an abelian extension \( K/\mathbb{Q} \), but the same proof gives the result relative to any abelian extension of number fields, cf. [Mai02, Proposition 19].)

Let \( S(B) \) denote the set of places of \( L^+ \) above \( v_0 \), \( R \) be as above, and \( S_l \) the set of places of \( L^+ \) dividing \( l \). Let \( v_1 \) denote the place of \( L^+ \) below \( \overline{v}_1 \), and let \( S_{\overline{v}_1} = \{v_1\} \). Let \( T = S = S(B) \cup S_l \cup S_{\overline{v}_1} \cup R \). We choose lifts of these sets to sets \( \overline{S}(B), \overline{S}_l, \overline{S}_{\overline{v}_1} \) and \( \overline{R} \) of places of \( L \), and set \( \overline{T} = \overline{S} = \overline{S}(B) \cup \overline{S}_l \cup \overline{S}_{\overline{v}_1} \cup \overline{R} \). With the above hypotheses, we can choose a definite unitary group \( \overline{G} \) over \( L^+ \) as in [4].

Since \( r_1(\pi) \) is irreducible and \( \overline{r}_1(\pi) \cong \overline{\mathcal{P}} \), we can choose an extension of \( \rho|_{G_{L,S}} \) to a homomorphism \( r : G_{L^+,S} \to \mathcal{G}_{\pi}(O) \) with \( r(c) \notin \mathcal{G}_p(k) \) and \( \nu \circ r = \epsilon^{1-n} \delta_{L/L^+}^\pi \). We may choose an extension of \( \rho|_{G_L} \) to a homomorphism \( r' : G_{L,S} \to \mathcal{G}_{\pi}(O) \) with \( \pi = \pi' \) and \( \nu \circ r' = \nu \circ r \).

We now have a deformation problem

\[
S = \left( L/L^+, T, \overline{T}, \Lambda, \pi, \epsilon^{1-n} \delta_{L/L^+}^\pi, \{R_v^\Lambda\}_{v \in S_l} \cup \{R_{\overline{v}_1}^\Lambda\}_{v \in S(B)} \cup \{R_v^{S_l}\}_{v \in \overline{S}_l} \cup \{R_v^\overline{T}\}_{v \in \overline{T}} \right),
\]

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and both \( r \) and \( r' \) are of type \( S \). The result will now follow from Theorem 6.1 above and [BLGHT11] Lemma 1.4 if we can show that the hypotheses of Theorem 6.1 are satisfied. Thus it remains to show that the quotient \( R_S^{\text{red}} \) of \( R_S^{\text{univ}} \) is finite over \( \Lambda \) and of dimension at most \( n[L^+ : \mathbb{Q}] - rn(n + 1) - 5 \). In fact, it suffices to show that for any minimal prime \( Q \subset R_S^{\text{red}}, R_S^{\text{red}}/Q \) is finite over \( \Lambda \) and of dimension at most \( n[L^+ : \mathbb{Q}] - rn(n + 1) - 5 \).

To show this we introduce the auxiliary deformation problems

\[
S_1 = \left( L/L^+, T, \bar{T}, A_1, \tau_1, \epsilon^{1-n} \delta_{L/L^+}, \{ R_v^\Delta \}_v \in \mathcal{S}_1 \cup \{ R_v^{\text{univ}} \}_v \in \mathcal{S}_n \cup \{ R_v^1 \}_v \in \mathcal{R}_S(B) \right),
\]

\[
S_2 = \left( L/L^+, T, \bar{T}, A_2, \tau_2, \epsilon^{1-n} \delta_{L/L^+}, \{ R_v^\Delta \}_v \in \mathcal{S}_1 \cup \{ R_v^{\text{univ}} \}_v \in \mathcal{S}_n \cup \{ R_v^1 \}_v \in \mathcal{R}_S(B) \right),
\]

where the local deformation problems are now taken with respect to the \( \tau_i \) (and we take the extensions to \( G_{n_i}(k) \)-valued representations induced by our choice of \( \tau \), \( n_i = \dim \mathfrak{p}_i \)). Here by definition we have \( \Lambda_i = \overline{\otimes}_{v \in \mathcal{S}_1} \Lambda_{v,i} = \overline{\otimes}_{v \in \mathcal{S}_1} \mathcal{O}[\pi_v^0(l)^n] \), as in (3.3). It follows from [Thor12] Corollary 8.7 and [BLGGT18] Lemma 1.2.2 that for \( i = 1, 2 \), \( R_S^{\text{univ}} \) is finite over \( \Lambda_i \) of dimension \( 1 + n_i[L^+ : \mathbb{Q}] \).

Let us write \( R = R_S^{\text{red}} / Q \) and \( r_Q : G_{L,S} \to G_0(R) \) for a lifting representing the induced deformation over \( R \). We may choose \( r_Q \) to be of the form \( r_Q = r_1 \oplus r_2 \), where \( r_j \) is a lift of \( \tau_j \). Let \( E = \text{Frac}(R) \), and choose an algebraic closure \( \mathcal{E} \). For each place \( v \in S \), we have the unrestricted lifting ring \( R_v^{\text{univ}} \) a \( \Lambda_v \)-algebra, and its quotient \( R_v^\Delta \). We recall (cf. 3.11) that there is a projective morphism \( G_{v} \to R_v^{\text{univ}} \), where \( G_v \) is \( \mathcal{O} \)-flat and reduced, and that \( R_v^\Delta \) is defined as the scheme-theoretic image of this morphism. In particular, the induced map \( R_v^\Delta \to R \) lifts to an \( E \)-point of \( G_v \), and hence there exists an increasing filtration

\[
0 = \text{Fil}_v^0 \subset \text{Fil}_v^1 \subset \cdots \subset \text{Fil}_v^n = r_Q|_{G_{L,S}} \otimes_R E \text{ of } r_Q|_{G_{L,S}} \otimes_R E \text{ with the property that the action of } I_{L,S} \text{ on } \text{gr}_v^i = \text{Fil}_v^i / \text{Fil}_v^{i-1} \text{ is given by the specialization of the universal character } \psi_v^i : I_{L,S} \to \Lambda_v^\times \text{ via the morphism } \Lambda_v \to R_v^\Delta \to E).
\]

We set \( F_v^i = \text{Fil}_v^i \cap (r_1|_{G_{L,S}} \otimes_R E) \) and \( G_v^i = \text{Fil}_v^i \cap (r_2|_{G_{L,S}} \otimes_R E) \). Then \( F_v^i \) and \( G_v^i \) are increasing filtrations with graded pieces of dimension at most one, and \( F_v^0 + G_v^0 = \text{Fil}_v^0 = r_Q|_{G_{L,S}} \otimes_R E \). We write \( \alpha_1^i, \ldots, \alpha_n^i \) for the characters \( I_{L,S} \to E^\times \) afforded by the non-trivial graded pieces \( \text{gr}^i F_v^i, i = 1, \ldots, n \), and \( \beta_1^i, \ldots, \beta_n^i \) for the characters afforded by the non-trivial graded pieces \( \text{gr}^i G_v^i \). Let us write \( \gamma_1^i, \ldots, \gamma_n^i \) for the characters \( \alpha_1^i, \ldots, \alpha_n^i, \beta_1^i, \ldots, \beta_n^i \). There exists a unique permutation \( \sigma_v \), increasing on \( \{1, \ldots, n\} \) and \( \{n_1 + 1, \ldots, n_1 + n_2\} \), such that \( \gamma_v^i \) is the specialization of the universal character \( \psi_v(\sigma_v(i)) \) via the morphism \( \Lambda_v \to E \).

The permutations \( \sigma_v \) define isomorphisms \( \Lambda_{v,1} \otimes \mathcal{O} \Lambda_{v,2} \cong \Lambda_v \) and \( \Lambda_{1} \otimes \mathcal{O} \Lambda_{2} \cong \Lambda \) in an obvious manner. Moreover, via these isomorphisms, \( R \) obtains the structure of a \( \Lambda_1 \)- and \( \Lambda_2 \)-algebra, and it makes sense to ask whether the liftings \( r_1, r_2 \) over \( R \) are of type \( S_1 \) or \( S_2 \), respectively. Let us introduce the further auxiliary deformation problems

\[
S_1' = \left( L/L^+, T, \bar{T}, A_1, \tau_1, \epsilon^{1-n} \delta_{L/L^+}, \{ R_v^\Delta \}_v \in \mathcal{S}_1 \cup \{ R_v^{\text{univ}} \}_v \in \mathcal{S}_n \cup \{ R_v^1 \}_v \in \mathcal{R}_S(B) \right),
\]

\[
S_2' = \left( L/L^+, T, \bar{T}, A_2, \tau_2, \epsilon^{1-n} \delta_{L/L^+}, \{ R_v^\Delta \}_v \in \mathcal{S}_1 \cup \{ R_v^{\text{univ}} \}_v \in \mathcal{S}_n \cup \{ R_v^1 \}_v \in \mathcal{R}_S(B) \right).
\]

It is clear that \( r_1 \) and \( r_2 \) are of type \( S_1' \) and \( S_2' \), respectively. To show that they are of type \( S_1 \) and \( S_2 \), it remains to show that for each prime \( v \in S_i \), the restrictions \( r_i|_{G_{L,S}} \) in fact define points of the quotient \( R_v^\Delta \) of \( R_v^{\text{red}} \). (We remark that we are abusing notation here by using the symbols \( R_v^\Delta \) and \( R_v^\Delta \) to denote lifting rings of representations of dimensions \( n_1, n_2 \) and \( n \).) However, this follows from the remark after Lemma 3.11. The induced homomorphism \( R_S^{\text{univ}} \otimes_R R_S^{\text{univ}} \to R \) is surjective, by universality, and is a homomorphism of \( \Lambda \)-algebras, by construction. Since the former ring here is a finite \( \Lambda \)-algebra, by the above, we deduce that \( R \) is also a finite \( \Lambda \)-algebra.

It remains to bound the dimension of \( R \). For \( i = 1, 2 \) let \( \psi_i : G_{L,S} \to O^\times \) be the Teichmüller lift of \( \det \tau_i|_{G_{L,S}} \). Write \( R_S^{\text{univ}} \), for the quotient of \( R_S^{\text{univ}} \), where the determinant of the universal deformation restricted to \( G_{L,S} \) is equal to \( \psi_i \). By Lemma 3.32, we have \( R_S^{\text{univ}} \cong R_S^{\text{univ}} \otimes \mathcal{O}[\Delta/(c+1)] \), and

\[
R_S^{\text{univ}} \otimes_R R_S^{\text{univ}} / \lambda \cong R_S^{\text{univ}} \otimes \mathcal{O}[\Delta/(c+1)] \otimes k[\Delta/(c+1)],
\]

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and the ring on the left hand side is flat over $k[[\Delta/(c + 1)]] \otimes_k k[[\Delta/(c + 1)]]^\times$ for the universal characters valued in this ring.

We claim that in the underlying reduced quotient ring of $R/\lambda$ the relation $\Psi_1(\text{Frob}_v)^n = \Psi_2(\text{Frob}_v)^n$ holds for each $v \in S(B)$. Indeed, if $p \subset R/\lambda$ is a prime ideal, then there exists $\alpha \in R/p$ such that $r_1(\text{Frob}_v)$ has characteristic polynomial $(X - \alpha)^{n_1}$ and $r_2(\text{Frob}_v)$ has characteristic polynomial $(X - \alpha)^{n_2}$. But then we have

$$\alpha^n = (\psi_1(\text{Frob}_v)\Psi_1(\text{Frob}_v)^{n_1})^{n_2} = (\psi_2(\text{Frob}_v)\Psi_2(\text{Frob}_v)^{n_2})^{n_1} \mod p,$$

and hence $\Psi_1(\text{Frob}_v)^n = \Psi_2(\text{Frob}_v)^n$. The quotient of the ring $k[[\Delta/(c + 1)]] \otimes_k k[[\Delta/(c + 1)]]^\times$ defined by the relations $\Psi_1(\text{Frob}_v)^n = \Psi_2(\text{Frob}_v)^n$ for $v \in S(B)$ has codimension at least $6 + rn(n + 1)$, by the choice of $L$; now the flatness shows that

$$\dim R \leq 1 + \dim R_{S_1}^{\text{univ}} \otimes_O R_{S_2}^{\text{univ}} / \lambda - (6 + rn(n + 1)) \leq n[L^+ : \mathbb{Q}] - rn(n + 1)/2 - 5,$$

as required. \hfill \Box

### 8 An application to the Fontaine-Mazur conjecture

We now combine our main theorem with Serre’s conjecture for $\text{GL}_2(\mathbb{A}_Q)$ to deduce the following result.

**Theorem 8.1.** Let $E/\mathbb{Q}$ be a quadratic imaginary extension and let $l$ be a prime. Suppose that $\rho : G_E \to \text{GL}_3(\mathbb{C}_l)$ is a continuous irreducible representation satisfying the following hypotheses:

(i) $\rho$ is ramified at only finitely many places.

(ii) $\rho^\vee \cong \rho^\vee e^{-2}$.

(iii) $\rho$ is crystalline ordinary of weight $\lambda$ for some $\lambda \in \{(\lambda_3,3)^{\text{Hom}(E,\mathbb{C}_l)}\}$. Moreover, $l$ splits in $E$ and for each embedding $\tau : E \to \mathbb{C}_l$, we have $\lambda_{\tau_1} > \lambda_{\tau_2} > \lambda_{\tau_3}$ and $6 + \sum^3_{j=1} (\lambda_{\tau_j} - \lambda_{\tau_3}) < l/2$.

(iv) There exists a place $v_0$ of $E$ split over $\mathbb{Q}$ and not dividing $l$ and an unramified character $\psi_0 : G_{E_{v_0}} \to \mathbb{C}_l^\times$ such that $\rho^{\text{ss}}$ is ramified at $v_0$ and $(\rho|_{G_{E_{v_0}}})^{\text{ss}} \cong \psi_0 \cong \psi_0 \circ \epsilon^2 \psi_0$. Moreover, $l$ does not divide $\prod^3_{i=1} (q_{v_0}^i - 1)$.

(v) $\overline{\rho^{\text{ss}}} = \overline{\rho}_1 \oplus \overline{\rho}_2$, where:

(a) $\dim \overline{\rho}_1 = 2$ and the image of $\overline{\rho}_1$ contains $\text{SL}_2(\mathbb{F}_l)$.

(b) $\dim \overline{\rho}_2 = 1$.

Then $\rho$ is automorphic, in the sense that it arises from a RACSDC automorphic representation of $\text{GL}_3(\mathbb{A}_E)$.

**Proof.** We fix an isomorphism $\iota : \mathbb{C}_l \cong \mathbb{C}$. Let $K \subset \mathbb{C}_l$ be a finite extension of $\mathbb{Q}_l$ over which $\rho$ is defined. We write, as usual, $\mathcal{O}$ for the ring of integers of $K$ and $k$ for its residue field. After possibly conjugating $\rho$ and enlarging $K$, we can extend $\rho$ to a continuous representation $r : G_{\overline{Q}} \to \mathcal{O}^\times$. Let $\mu = \nu \circ r : G_{\overline{Q}} \to \mathcal{O}^\times$.

If $c \in G_{\overline{Q}}$ is a complex conjugation and $r(c) = (A, -\mu(c))j$, then $\text{Tr} A = -\mu(c)A$. Since $3$ is odd, $A$ must be symmetric and therefore $\mu(c) = -1$.

Reducing modulo the maximal ideal of $\mathcal{O}$, we see that there is an extension of $\overline{\rho}_1$ to a homomorphism $r_1 : G_{\overline{Q}} \to G_2(k)$ such that if $\mu_1 = \nu \circ r_1$ then $\mu_1(c) = -1$. Let $\chi = \det \overline{\rho}_1$. Then $\overline{\chi} = \overline{\chi}^{-1} = \overline{\epsilon}^{-4}$. Since $\dim \overline{\rho}_1 = 2$, we have $\overline{\rho}_1 \cong \overline{\rho}_1^\vee$. Since $(\overline{\chi}^2)/(\overline{\epsilon}^2)^c = 1$, we can find a character $\overline{\psi} : G_E \to \mathbb{F}_l^\times$ such that $\overline{\psi}^e/\overline{\psi} = \overline{\chi}^2\overline{\epsilon}^2$. The representation $\overline{\rho}_1 \overline{\psi}$ now satisfies

$$(\overline{\rho}_1 \overline{\psi})^e \cong \overline{\rho}_1^\vee \overline{\epsilon}^{-2} \overline{\psi} \cong \overline{\rho}_1 \overline{\chi}^{-1} \overline{\epsilon}^{-2} \overline{\psi} \cong \overline{\rho}_1 \overline{\psi}.$$ 

Thus the representation $\overline{\rho}_1 \overline{\psi}$ extends to a continuous representation $\overline{\mathbf{1}}_1 : G_{\overline{Q}} \to \text{GL}_2(\mathbb{F}_l)$. In fact, this representation is odd (det $\overline{\mathbf{1}}_1(c) = -1$): the self-duality of $\overline{\mathbf{1}}_1$ is symplectic, the conjugate self-duality of $\overline{\rho}_1 \overline{\psi}$
is orthogonal, and an easy calculation shows that the difference of these signs is given by $\det \overline{R}_1(c)$. (We learned this observation from Frank Calegari.)

Fix an embedding $\tau : E \hookrightarrow \overline{Q}_l$, and let $v$ denote the induced place of $E$ above $l$. Since $\rho$ is crystalline of weight $\lambda$, we have

$$\rho|_{I_{E_v}} \cong \begin{pmatrix} \varepsilon^{-\lambda_3} & * & * \\ 0 & \varepsilon^{-(\lambda_2 + 1)} & * \\ 0 & 0 & \varepsilon^{-(\lambda_1 + 2)} \end{pmatrix},$$

and hence

$$\overline{R}_1|_{I_{E_v}} \cong \overline{\psi} \begin{pmatrix} \varepsilon^{a} & * \\ 0 & \varepsilon^{b} \end{pmatrix},$$

for some $a < b$ in $\{\lambda_3, \lambda_2 + 1, \lambda_1 + 2\}$. Let $c$ denote the other element of $\{\lambda_3, \lambda_2 + 1, \lambda_1 + 2\}$. By Serre’s conjecture, $\overline{R}_1$ is residually automorphic.

Let $E_0$ be an imaginary quadratic field in which $l$ splits, the prime below $v_0$ is inert, and in which every other prime below a place of $E$ at which $\rho$ or $E$ is ramified splits. Let $E_1 = E_0 \cdot E$. Then $E_1/E_1^\perp$ is an everywhere unramified quadratic extension, split at every prime at which $\rho$ is ramified. By [BLGG] Theorem A and [BLGGL], we can find an $\iota$-ordinary RAECSDC automorphic representation $\pi_1$ of $GL_2(\mathbb{A}_{E_1})$ such that $\pi_1^\perp \cong \pi_1^! \cdot r_{\iota}(\pi_1)$ and $r_{\iota}(\pi_1)$ is crystalline with $HT_{\iota}(r_{\iota}(\pi_1)) = \{a, b\}$, for any embedding $\iota : E_1 \hookrightarrow \overline{Q}_l$ such that $\iota(\pi_1)|_E = \tau$. We may further suppose that $\pi_{1, v_0}$ is an unramified twist of the Steinberg representation. Let $\rho_1 = r_{\iota}(\pi_1)$.

We can also choose a lift of $\pi_2$ to a character $\rho_2 : G_{E_1} \to \overline{Q}_l$ satisfying $\rho_2^\perp = \rho_2^! \cdot \varepsilon^2$, unramified at $v_0$, and such that $\rho_2$ is crystalline with $HT_{\iota}(\rho_2) = \{c\}$. Let $\pi_2$ denote the RAECSDC automorphic representation of $GL_1(\mathbb{A}_{E_1})$ corresponding under $\iota$ to $\rho_2|_{G_{E_1}}$. Then $\pi_2^\perp \cong \pi_2^! \cdot \varepsilon^2$.

The automorphic representation $\Pi = \pi_1 \cdot \varepsilon^{1/2} \oplus \pi_2 \cdot \varepsilon^{-1}$ is regular algebraic and conjugate self-dual, and satisfies $r_{\iota}(\Pi) = \rho_1 \oplus \rho_2$. (We have not mentioned here the Galois representation associated to a regular algebraic and conjugate self-dual but not necessarily cuspidal automorphic representation, but it exists and satisfies the properties in Theorem 2.1 [see [Tho] Theorem 2.1].) By [Tho] Theorem 7.1, there exists an $\iota$-ordinary RACSDC automorphic representation $\pi$ of $GL_2(\mathbb{A}_{E_1})$ such that $r_{\iota}(\pi) \cong \overline{\psi} \cdot \pi_1^{|} \cdot \varepsilon^2$ and $\pi$ is an unramified twist of the Steinberg representation at a place of $E_1$ above $v_0$. Theorem 7.1 now applies to $\rho|_{G_{E_1}}$, and the result follows by soluble base change. 

References


