DIRICHLET'S PRINCIPLE AND PROBLEM

PATRICK RYAN

1. Introduction

This is a topic rich in physical application, history, and mathematical complexity. I will endeavor to give the reader a brief survey of these areas, in a style that incorporates both mathematics and history. I will begin with the motivating force of potential theory, and proceed to describe the role of Dirichlet’s Principle in complex function theory (due to Riemann), critiques of the principle, and finally its revival in the modern era of mathematics. The amount of information on this topic is enormous, thus this overview will be somewhat cursory, and leaves the more precise and intricate arguments to the interest of the reader; however, some degree of mathematical sophistication is assumed. References are provided for those who wish to learn more. First, I will make an essential semantic distinction, which has caused great confusion in the past. When I speak of Dirichlet’s Problem, I mean the problem of determining a harmonic function in the interior of a domain. More specifically, let \( \Omega \) be an open subset of \( \mathbb{R}^3 \), and let there be a constant real-valued function \( f \) on \( \partial \Omega \), the boundary of \( \Omega \). The problem is finding some function \( F \), defined on the closure \( \bar{\Omega} \) of \( \Omega \), harmonic in \( \Omega \), and such that \( f = F \) on \( \partial \Omega \). Dirichlet’s Principle, which will be stated explicitly later, is a method of resolving this problem.

2. History

2.1. Dirichlet the Man. Born in 1805 to a family of Belgian descent living in Aachen, Johann Peter Gustav Lejeune Dirichlet took an early interest in science and mathematics. Inspired by Gauss’s Disquisitiones Arithmeticae, Dirichlet traveled to Paris at the age of 16 to audit lectures at the Faculte des Sciences. Soon, he was employed as a tutor to the General Maximilien Sebastien Foy. After Foy’s death, Alexander von Humboldt, the famous Prussian geographer and explorer, recruited Dirichlet to return to Germany and helped him obtain a professorship at the University of Berlin. There Dirichlet developed a relationship with Karl Jacobi and Jakob Steiner, and eventually married Rebecca Mendelssohn, the sister of composer Felix Mendelssohn.

In 1847, Riemann traveled to Berlin to study with Dirichlet and his mathematical circle. After two years, Riemann returned to Göttingen to complete his studies with Gauss, and in 1851, published his doctoral thesis, “The Foundations for a General Theory of Functions of a Complex Variable Magnitude,” in which he applied Dirichlet’s Principle, but did not ascribe it to Dirichlet. After Gauss’s death in 1855, Dirichlet accepted his chair at Göttingen, and continued his association with Riemann. Finally, in his seminal paper “The Theory of Abelian Functions”, Riemann identified for the first time “Dirichlet’s Principle.” Two years later, Dirichlet died, and Riemann assumed his position at Göttingen.
2.2. Potential Theory and the Equilibrium Problem. In short, we may describe potential theory as the study of harmonic functions, that is functions satisfying the Laplace equation (†) given below. The term arises from the fact that, in 19th century physics, the fundamental forces of nature were believed to be derived from potentials. Today, the situation is acknowledged as more complicated, with the Laplace equation being only a limiting case. However, the name remains as a convenient term for describing functions that satisfy the Laplace equation and its generalizations. Potential theory has its origins in Newtonian mechanics, and eventually spread into problems of magnetism and electricity. The first statement of what came to be called “Dirichlet’s Problem” was given by Gauss in the 1830s in the context of a potential theoretic problem. Primarily, Gauss concerned himself with the equilibrium problem, or determining the distribution of electric charges on a surface $S$ such that the potential is constant on the surface. Gauss obtained a solution by taking a potential

$$V = \int \int_S \frac{\rho}{r} ds$$

and any continuous function $U$ on $S$, and considered a family of integrals

$$\Omega = \int \int_S (V - 2U) \rho ds,$$

and attempted to show, that if a $\rho > 0$ is selected in such a way that $\Omega$ has a minimum, then the potential $V - U$ is constant on the surface $S$. Gauss claimed that the existence of this $\rho$ is evident, a reasonable statement, as the density function $\rho$ clearly satisfies Laplace’s equation outside of the given surface. Later, Dirichlet claimed that the existence of his minimizing function was evident; however, he was required to demonstrate that it satisfied Laplace’s equation, a much more complicated task without the assistance of physical intuition.

2.3. Dirichlet’s Principle. In 1856-57 Dirichlet lectured on potential theory in Göttingen. After his death, his lectures were published by F. Grube in 1876 as the first textbook on potential theory. It introduced the concept of the potential, its properties (namely Laplace’s equation), and applications to physical problems. Here is the original statement of Dirichlet’s Principle from this publication:

For every bounded, connected domain $T$ there are infinitely many functions $u$ continuous together with their first-order derivatives, which reduce to a given value on this surface. Among this class of functions, there will be at least one which reduces the following integral

$$U = \int \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2 \right] dT,$$

extended over the domain $T$, to a minimum. It is evident that this integral has a minimum, since it cannot be negative. We now show the following conditions:

(i) Every such minimizing function $u$, satisfies the differential equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad (\dagger)$$
everywhere in the domain $T$. This already makes it clear that there is always a function $u$ having the desired property, namely a function for which $U$ becomes a minimum.

(ii) Every function $u$ which satisfies the differential equation within the domain $T$, minimizes $U$.

(iii) The integral $U$ can have only one minimum.

It follows from conditions (ii) and (iii) that there is only one function $u$ with the desired property (Dirichlet, 1876, pp. 127-8).

Clearly, there are some claims here that must be justified, and served as the basis for later critiques. Dirichlet observed that it is evident that there exists at least one admissible function for which $U$ is minimized because the integral is non-negative for any $u$ belonging to the family of functions. He made the error of concluding that the greatest lower bound of $U$ is attained for some function in the family, i.e. that the lower bound is a minimum. This error cast a great deal of doubt upon his method. Before examining critiques of Dirichlet’s Principle, I will give a brief summary of its most important application to pure mathematics.

2.4. **Riemann’s Complex Function Theory.** While Gauss and Dirichlet studied potential theory as a field unto itself, Riemann used Dirichlet’s Principle, and results of potential theory, to get results for analytic functions. The method of minimizing a integral was used by Riemann in his two fundamental papers, “The Foundations for a General Theory of Functions of a Complex Variable Magnitude,” and “The Theory of Abelian Functions.” By means of decomposing a function into its real and imaginary parts, Riemann reduced the theory of complex analytic functions to the study of harmonic functions. Riemann applied potential theory to $\mathbb{R}^2$ and not $\mathbb{R}^3$, so he considered Dirichlet’s integral for two variables

$$I = \int \int \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right) dxdy,$$

thereby reducing the existence of a harmonic function with given values on the boundary (Dirichlet’s Problem) to the problem of minimizing $I$ when $u$ belongs to a suitable family of functions. Like Gauss and Dirichlet, Riemann observed that $I$ is never negative, and thus concluded that there must exist an admissible function for which it attains its greatest lower bound. Unfortunately, after this great achievement by Riemann using Dirichlet’s Principle, many mathematicians concluded that there was something decidedly wrong with the method. As a result, Riemann’s results came under attack, and many sought different methods of proof.

3. **Critiques**

3.1. **Weierstrass.** Riemann’s theory on the existence of complex analytic functions is based upon the solution of Dirichlet’s problem. He reduced this to the problem of minimizing an integral. However, this method was insufficient: Riemann failed to show that the Dirichlet integral $I$ attains its greatest lower bound in the set of harmonic functions. It was later shown that there exist open sets for which Dirichlet’s problem has no solution, namely there are open sets for which there is not a harmonic function contained in the set, which takes arbitrary boundary values. Conversely, for these sets, the minimum problem has no solution.
The most famous critique of Dirichlet’s Principle was proposed by Weierstrass in 1870, which I will state briefly here. Consider the family $\mathcal{F}$ of continuous functions $\psi$ defined on $[-1, 1]$, which are continuously differentiable on $(-1, 1)$, verifying $\psi(-1) = a, \psi(1) = b$, for given real numbers $a, b$ and $a \neq b$. Consider the family of integrals (with $\psi \in \mathcal{F}$)

$$J = \int_{-1}^{1} \left( x \frac{d\psi}{dx} \right)^2 dx$$

A brief computation verifies that $\inf_{\psi} J(\psi) = 0$. However, there is no such function in the family such that $J(\psi) = 0$. For such a function, one should have $d\psi/dx = 0$ for all $x \in (-1, 1)$, because the derivative of any function in the family is supposed to be continuous. Thus, $\psi$ would be a constant, however we have $\psi(-1) \neq \psi(1)$ for any admissible function.

Beyond any technical consideration, this shows that there are variational problems which do not have a solution, and casts severe doubt upon Dirichlet’s Principle. As a result, many mathematicians were suspicious of Riemann’s results in complex analysis. However, Riemann’s results were verified more rigorously by Carl Neumann [1877] and Schwarz [1872]. Thus, as Felix Klein stated in 1894, “We must conclude that Riemann originally derived the theorems themselves from physical intuition, which here again proved its value as a heuristic principle, and only afterwards based it on Dirichlet’s Principle in order to have an entirely mathematical train of thought.”

4. THE REVIVAL OF DIRICHLET’S PRINCIPLE

Weierstrass’s critique constrained mathematicians to do without Dirichlet’s Principle as a demonstrative argument for many years. However, the rigor of Weierstrass held much less sway over the community of physicists. Helmholtz (1821-1894), summarized this point of view, “...for we physicists, Dirichlet’s Principle continues to remain a demonstration.” This clearly distinguished the proofs of physicists from the proofs of mathematicians, for on the whole physicists are much less concerned with the mathematical details and will accept the evidence as sufficient.

Nonetheless, Dirichlet’s Principle came to be used by mathematicians at the end of the 19th century, most notably by Poincaré, and those active in the new school of functional analysis. Finally in 1900-1901, Hilbert published two papers which effectively revived Dirichlet’s Principle. In his work, he considered open sets for which the boundary consists of analytic curves and the boundary values are analytic, thereby proving the correctness of Dirichlet’s Principle. He stated that Dirichlet’s problem is only a special case in the calculus of variations, and hence we arrive at a statement of Dirichlet’s Principle in this more general form: “Every regular problem of the calculus of variations has a solution, provided restrictive assumptions regarding the nature of the given boundary conditions are fulfilled and, when necessary, the concept of solution has been suitably extended” (Hilbert, 1900, p.11). With Hilbert’s re-statement and defense of Dirichlet’s Principle, it was inaugurated into the modern era of mathematical research.
5. Bibliography


