The universal cover of $SO(n)$

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1 Introduction

Given a vector space $V$ and a non-singular quadratic form $Q$ on $V$, the orthogonal group $O(Q)$ is the subgroup of $GL(V)$ that preserves $Q$. The special orthogonal group $SO(Q)$ is given by $O(Q) \cap SL(V)$.

If $Q$ is the standard inner product on $\mathbb{R}^n$ then $SO(Q)$ is denoted $SO(n)$. This is a connected, compact Lie group. For $n > 2$, the group is semi-simple. In the case that $n$ is odd, $SO(n)$ corresponds to the Dynkin diagram $B_n$. In the case that $n$ is even, $SO(n)$ corresponds to the Dynkin diagram $D_n$.

The special orthogonal group $SO(n)$ is unique among the classical groups in not being simply connected. To study the representations of its Lie algebra $so(n)$, it is necessary to understand the universal cover of $SO(n)$. The project of this paper is to compute the root system of $SO(n)$, to show that $SO(n)$ is not simply connected by determining its fundamental group, and to explicitly construct its universal cover when $n > 2$. The universal cover, $Spin(n)$, is constructed as a particular subgroup of the Clifford algebra of $Q$. Clifford algebras are defined and discussed in section 4.1 to prepare for the construction of $Spin(n)$ in section 4.2.
2 The root-system of $\text{SO}(n)$

Directly computing the root system of $\text{SO}(n)$ is messy, so we will introduce an isomorphic Lie group whose matrices are easier to work with. We define a quadratic form $E_n$ on $\mathbb{C}^n$ via the matrix:

$$E_{2m} = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix},$$

$$E_{2m+1} = \begin{pmatrix} E_{2m} & 0 \\ 0 & 1 \end{pmatrix}.$$ 

We define $\text{SO}(E_n)$ to be the set of matrices in $\text{SL}(n, \mathbb{C})$ that preserve the form $E_n$ and that are real with respect to it:

$$\text{SO}(E_n) \equiv \{ g \in \text{SL}(n, \mathbb{C}) | g = E_n g E_n, g^T E_n g = E_n \}.$$ (1)

This is a Lie group. Its Lie algebra is given by

$$\mathfrak{so}_\mathbb{R}(E_n) = \{ X \in M(n, \mathbb{C}) | X = E_n X E_n, X^T E_n + E_n X = 0 \}$$ (2) and its complexified Lie algebra is given by

$$\mathfrak{so}(E_n) = \{ X \in M(n, \mathbb{C}) | X^T E_n + E_n X = 0 \}.$$ (3)

The isomorphism from $\text{SO}(n)$ to $\text{SO}(E_n)$ is realized by $g \mapsto T_n^{-1} g T_n$ where

$$T_{2m} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_m & I_m \\ iI_m & -iI_m \end{pmatrix},$$

$$T_{2m+1} = \begin{pmatrix} T_{2m} & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Thus we can understand the root structure of $\text{SO}(n)$ by understanding the root structure of $\text{SO}(E_n)$. We begin by computing the Lie algebra. The conditions in equation 2 imply that $\mathfrak{so}(E_{2m})$ consists of matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

such that $B$ and $C$ are each skew-symmetric and $A = -D^T$. The real Lie algebra $\mathfrak{so}(E_{2m})$ has the additional conditions that $A = D$ and $B = C$.

**Remark 2.1.** It follows that $\text{SO}(E_2) \cong \mathbb{C}^*$. Note that this Lie group is not semi-simple. In this special case of $n = 2$, $\pi_1(\text{SO}(n)) = \mathbb{Z}$ and $\text{Spin}(n)$ is not the universal cover of $\text{SO}(n)$. For the remainder of the paper, we will deal with $n \geq 3$.

Computing the Lie algebra in the odd case is a little messier. Consider $X \in \mathfrak{so}(E_{2m+1})$ as a block matrix of the form

$$\begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{pmatrix}$$
where $A_1 \ldots A_4$ are $m \times m$, $A_5$ and $A_6$ are $m \times 1$, $A_7$ and $A_8$ are $1 \times m$ and $A_9$ is $1 \times 1$. Then the conditions on $A_1 \ldots A_4$ are identical to the conditions on $A_7 \ldots D$ in the even case. There are the further conditions that $A_5 = -A_7^T$, $A_6 = -A_5^T$ and $A_9 = 0$. The real Lie algebra $so(E_{2m+1})$ has the additional conditions: $A_7 = \overline{A}_8$ and $A_5 = \overline{A}_6$.

Then Cartan subalgebras (maximal Abelian subalgebras) for both Lie algebras are given by

$$t_R(E_{2m}) = \{ \text{diag}(i\theta_1, \ldots, i\theta_m, -i\theta_1, \ldots, -i\theta_m) | \theta_i \in \mathbb{R} \},$$
$$t_R(E_{2m+1}) = \{ \text{diag}(i\theta_1, \ldots, i\theta_m, -i\theta_1, \ldots, -i\theta_m, 0) | \theta_i \in \mathbb{R} \}.$$

These spaces have the same dimension, and if we denote the standard basis of $M(n, \mathbb{C})$ by $M_{i,j}$ (suppressing the $n$) then the basis for both spaces can be written as

$$\{ B_j = i(M_{j,j} - M_{j+m,j+m}) | j \in \{1, \ldots, m\} \}.$$

The dual bases for $it_R^*$ will be denoted

$$\{ L_1, \ldots, L_m | \langle L_j, B_k \rangle = i\delta_{jk} \forall k \in \{1, \ldots, m\} \}.$$

The complexifications of the Cartan algebras are:

$$t(E_{2m}) = \{ \text{diag}(z_1, \ldots, z_m, -z_1, \ldots, -z_m) | z_i \in \mathbb{C} \},$$
$$t(E_{2m+1}) = \{ \text{diag}(z_1, \ldots, z_m, -z_1, \ldots, -z_m, 0) | z_i \in \mathbb{C} \}.$$

**Proposition 2.2** (Roots of $SO(2n)$). The set of roots of $SO(E_{2n})$ is

$$\Phi = \{ \pm(L_i \pm L_j) | 1 \leq i < j \leq n \}.$$

The corresponding root spaces are

$$so(2n)^{L_i-L_j} = \mathbb{C}(M_{i,j} - M_{j+n,i+n}),$$
$$so(2n)^{L_i+L_j} = \mathbb{C}(M_{j,i} - M_{i+n,j+n}),$$
$$so(2n)^{L_j} = \mathbb{C}(M_{i,j+n} - M_{j,i+n}),$$
$$so(2n)^{-L_j} = \mathbb{C}(M_{i+n,j} - M_{j+n,i}).$$

Denoting $M_{i,i}$ by $M_i$, the corresponding $H_\alpha$ are

$$H_{L_i-L_j} = (M_i - M_j) - (M_{i+n} - M_{j+n}),$$
$$H_{L_i+L_j} = (M_i - M_j) - (M_{i+n} + M_{j+n}),$$

and the values for the negative weights given by $H_{-\alpha} = -H_\alpha$.

**Proposition 2.3** (Roots of $SO(2n+1)$). The set of roots of $SO(E_{2n+1})$ is

$$\Phi = \{ \pm(L_i \pm L_j) | 1 \leq i < j \leq n \} \cup \{ \pm L_i | 1 \leq i \leq n \}.$$
The corresponding root spaces are
\[
\begin{align*}
\text{so}(2n+1)^{L_i-L_j} &= \mathbb{C}(M_{i,j} - M_{j+n,i+n}), \\
\text{so}(2n+1)^{-L_i+L_j} &= \mathbb{C}(M_{i,i} - M_{i+n,j+n}), \\
\text{so}(2n+1)^{L_i+L_j} &= \mathbb{C}(M_{i,j+n} - M_{j,i+n}), \\
\text{so}(2n+1)^{-L_i-L_j} &= \mathbb{C}(M_{i+n,j} - M_{j+n,i}), \\
\text{so}(2n+1)^{L_i} &= \mathbb{C}(M_{i,2n+1} - M_{2n+1,i+n}), \\
\text{so}(2n+1)^{-L_i} &= \mathbb{C}(M_{i+n,2n+1} - M_{2n+1,i}).
\end{align*}
\]

The corresponding $H_\alpha$ are
\[
\begin{align*}
H_{L_i-L_j} &= (M_i - M_j) - (M_{i+n} - M_{j+n}), \\
H_{L_i+L_j} &= (M_i + M_j) - (M_{i+n} + M_{j+n}), \\
H_{L_i} &= 2M_i - 2M_{i+n},
\end{align*}
\]
and the corresponding values for the negative weights.

3 The fundamental group of $\text{SO}(n)$

We compute the fundamental group of $\text{SO}(n)$ using the following result from class:

**Proposition 3.1.** Let $G$ be a connected, compact, and semisimple Lie group. The unit lattice of the universal cover of $G$ is given by
\[
L_{\text{sc}} = \text{span}_\mathbb{Z}\{2\pi i H_\alpha | \alpha \in \Psi\}
\]
and the fundamental group of $G$ is a quotient of $L/L_{\text{sc}}$ where $L$ is the unit lattice of $G$.

From the computation of the previous section, we see that
\[
L(\text{SO}(2m)) = \text{span}_\mathbb{Z}\{2\pi i (M_j - M_{j+m}) | j \in \{1, \ldots, m\}\}
\]
\[
L_{\text{sc}}(\text{SO}(2m)) = \left\{ \text{diag}(i\theta_1, \ldots, i\theta_m, -i\theta_1, \ldots, -i\theta_m) \sum \theta_i \in 2\mathbb{Z} \right\}
\]
and
\[
L(\text{SO}(2m+1)) = \text{span}_\mathbb{Z}\{2\pi i (M_j - M_{j+m}) | j \in \{1, \ldots, m\}\}
\]
\[
L_{\text{sc}}(\text{SO}(2m+1)) = \left\{ \text{diag}(i\theta_1, \ldots, i\theta_m, -i\theta_1, \ldots, -i\theta_m, 0) \sum \theta_i \in 2\mathbb{Z} \right\}.
\]

In each case, the quotient $L/L_{\text{sc}}$ is $\mathbb{Z}/2\mathbb{Z}$. The implication is that $\text{SO}(n)$ is not its own universal cover. Thus it is not simply connected, and we have:

**Proposition 3.2.** The fundamental group of $\text{SO}(n)$ is given by
\[
\pi_1(\text{SO}(n)) = \mathbb{Z}/2\mathbb{Z}
\]
the cyclic group of order 2.

It follows from this proposition that the universal cover of $\text{SO}(n)$ is a covering of order 2.
4 Explicitly constructing Spin$(n)$

4.1 Clifford algebras

Let $V$ be a vector space over the field $F$. We denote the tensor algebra by

$$T(V) = \bigoplus_{n \geq 0} V^\otimes n = \mathbb{C} \oplus V \oplus (V \otimes V) \oplus \ldots.$$ 

Let $Q$ be a quadratic form on $V$.

**Definition 4.1.** The Clifford algebra of $Q$ is the quotient algebra $T(V)/J$ where $J$ is the ideal of $T(V)$ generated by $v \otimes v - Q(v)$.

Many mathematicians make the opposite sign choice in this definition, but the theory is fundamentally the same in either convention.

We will denote the Clifford algebra by $\text{Cl}_Q(V)$. It has a privileged subspace that is isomorphic to $V$. The subalgebra of units of $\text{Cl}_Q(V)$ will be denoted $\text{Cl}_Q(V)^*$. Suppose, for the remainder of this paper, that the characteristic of $F$ is not 2. Most of the theory of Clifford algebras is developed only in this case. Then the defining identity is equivalent to:

$$uv + vu = 2(u, v)Q \forall u, v \in V$$

where $(u, v)_Q = \frac{1}{2}(Q(v + w) - Q(v) - Q(w))$.

The following remark indicates the significance of Clifford algebras:

**Remark 4.2 (Universal property).** The Clifford algebra $\text{Cl}_Q(V)$ is the universal algebra with the following property: Given an associative, unital algebra $E$ and a linear map $j : V \to E$ such that $j(v)^2 = Q(v)1_E$ for all $v \in V$, then there is a unique homomorphism of algebras $f : \text{Cl}_Q(V) \to E$ extending $j$.

For our purpose, it suffices to understand the center and two important involutions of the Clifford algebra. We will assume, for the rest of the paper that $Q$ is non-degenerate.

**Lemma 4.3.** The center of $\text{Cl}_Q(V)$, denoted $Z_{\text{Cl}_Q(V)}$, is $F$ if $\dim V$ is even. If $\dim V$ is odd then the center is $F + cF$ where $c = e_1 e_2 \ldots e_{\dim V}$ for $e_i$ an orthonormal basis of $V$.

**Proof.** Let $e_1, \ldots, e_n$ be an orthonormal basis for $V$. We have the defining commutation relations: $e_i^2 = Q(e_i) = 1$ and $e_i e_j = -e_j e_i$ for $i \neq j$. For $I = (i_1, \ldots, i_k)$ an ordered subset of $\{1, \ldots, n\}$ without repetition, let $e_I = e_{i_1} \ldots e_{i_k}$. Note that:

$$e_I e_J = \begin{cases} (-1)^{|I||J|} e_I e_J & \text{if } j \notin I, \\ (-1)^{|I|-1} e_I e_J & \text{if } j \in I. \end{cases}$$
Thus \( e_I \in Z_{\text{Cl}_Q(V)} \) if \(|I|\) is even and \( I \) is empty or if \(|I|\) is odd and every \( j \) is in \( I \).

There are two important involutions of \( \text{Cl}_Q(V) \). The first is:

**Definition 4.4.** The **grade involution** \( \alpha_{\text{Cl}_Q(V)} \) of \( \text{Cl}_Q(V) \) is the involution induced by the linear map \( v \mapsto -v \) on \( V \).

This is well-defined because the map \( v \mapsto -v \) preserves \( Q \). The \(+1\) and \(-1\) eigenspaces of \( \alpha_{\text{Cl}_Q(V)} \) are denoted \( \text{Cl}_Q^+(V) \) and \( \text{Cl}_Q^-(V) \) respectively. The subalgebra \( \text{Cl}_Q^+(V) \) is spanned by products \( v_1 v_2 \ldots v_{2k} \) of an even number of vectors \( v_i \in V \). The subalgebra \( \text{Cl}_Q^-(V) \) by products of an odd number of vectors.

**Definition 4.5.** The **transpose involution** of \( \text{Cl}_Q(V) \) is induced by the transpose involution on \( T(V) \). We denote it \( w \mapsto w^* \).

This is well-defined, because transposition preserves the ideal \( J \subset T(V) \). The transpose involution takes the product \( v_1 \ldots v_m \) of vectors \( v_i \in V \) to the product \( v_m \ldots v_1 \). It is the identity on \( V \subset \text{Cl}_Q(V) \) and preserves \( \text{Cl}_Q^+(V) \) and \( \text{Cl}_Q^-(V) \). We will refer to the transpose involution as the **principal involution**.

Given our sign-choice in defining the Clifford algebra, it is the most “significant” involution in a sense that will become clear.

### 4.2 Spin(\( n \))

Having stated several generalities about the Clifford algebra, we are prepared to develop the spin groups. Recall that \( \text{Cl}_Q(V) \) is the Clifford algebra for a non-degenerate quadratic form over a field of characteristic not equal to two.

We begin by defining the two Clifford groups:

\[
\Gamma(V, Q) = \{ x \in \text{Cl}_Q(V)^* | xv x^{-1} = V \}
\]

\[
\Gamma^+(V, Q) = \Gamma(V, Q) \cap \text{Cl}_Q^+(V).
\]

Given \( x \in \Gamma(V, Q) \) and \( v \in V \), we have:

\[
Q(xvx^{-1}) = (xvx^{-1})^2 = xv^2 x^{-1} = Q(v).
\]

Therefore the map \( v \mapsto xvx^{-1} \) is an orthogonal transformation of \( V \) and this correspondence gives a homomorphism \( \pi : \Gamma(V, Q) \to O(Q) \). If \( v, w \in V \) and \( Q(v) \neq 0 \) then

\[
v^{-1} = \frac{v}{Q(v)}.
\]

Then we have

\[
vwv^{-1} = \frac{2(v, w)Q}{Q(v)} v - w.
\]

Thus for \( v \in \Gamma(V, Q) \), \( \pi(v) = -r_v \) where \( r_v \in O(Q) \) is reflection across the hyperplane orthogonal to \( v \). We then apply the following theorem:
Theorem 4.6 (Cartan-Dieudonné). Let \((V, B)\) be an \(n\)-dimensional, non-degenerate, symmetric bilinear space over a field \(F\) of characteristic \(\neq 2\). Then every element of the orthogonal group \(O(B)\) is a composition of at most \(n\) reflections.

See [2] for proof and discussion.

In our case, this theorem says that every element of \(SO(Q)\) is the product of an even number of reflections and every improper element of \(O(Q)\) is the product of an odd number of reflections. This is the key ingredient for:

Lemma 4.7. If \(\dim V\) is even, \(\pi : \Gamma(V, Q) \to O(Q)\) is surjective. If \(\dim V\) is odd, then the image is \(SO(Q)\).

Proof. Cartan-Dieudonné proves that \(\pi\) is surjective in the even case and that its image always contains \(SO(Q)\). All that remains to prove is that \(\pi\) is not surjective in the odd case. If this were true then there is \(x \in \Gamma(V, Q)\) such that \(xv x^{-1} = -v\) for all \(v \in V\). However, this is a contradiction by proposition 4.3 because this implies that \(x\) acts non-trivially on \(e\).

The elements of \(\ker \pi\) commute with the elements of \(V\). The elements of \(V\) generate \(Cl_Q(V)\), so \(\ker \pi\) is \(Z_{Cl_Q(V)}^* \equiv Z_{Cl_Q(V)} \cap Cl_Q(V)^*\). We have the exact sequence:

\[ 1 \to Z_{Cl_Q(V)}^* \to \Gamma(V, Q) \xrightarrow{\pi} O(Q). \]

We wish to consider the restriction of \(\pi\) to \(\Gamma^+(V, Q)\). The above sequence gives us that \(x \in \Gamma(V, Q)\) can be written as \(\alpha v_1 \ldots v_k\) where \(\alpha \in Z_{Cl_Q(V)}^*\). If \(\dim V\) is odd then the image of \(\pi\) is \(SO(Q)\), so we can take \(k\) to be even. Then if \(x \in \Gamma^+(V, Q)\), we get that \(\alpha \in F^*\). If \(\dim V\) is even then \(\alpha \in F^*\), so if \(x \in \Gamma^+(V, Q)\) then \(k\) is even. In either case, we have the exact sequence

\[ 1 \to F^* \to \Gamma^+(V, Q) \xrightarrow{\pi} SO(Q) \to 1. \]

Now we introduce the map \(N(w)\) on \(Cl_Q(V)\) defined by \(N(w) = ww^*\) (the star denotes the action of the principal involution defined in 4.5). We have:

Lemma 4.8. The map \(N(w)\) gives a homomorphism \(\Gamma^+(V, Q) \to F^*\).

Proof. It was proven above that any element \(w \in \Gamma^+(V, Q)\) is given by \(\alpha v_1 \ldots v_k\) for \(\alpha \in F^*\). It maps to

\[ \alpha Q(v_1) \ldots Q(v_k) \]

and its product with \(\beta u_1 \ldots u_j\) maps to

\[ \alpha \beta Q(v_1) \ldots Q(v_k)Q(u_1) \ldots Q(u_j). \]

Now we give the spin group:
Definition 4.9. The spin group of the quadratic form \( Q \) is given by

\[
\text{Spin}(Q) = \{ w \in \Gamma^+(V,Q) | N(w) = 1 \}.
\]

For the remainder of this paper, we assume that we are in one of two cases:

1. \( F = \mathbb{R} \) and \( Q \) is definite
2. \( F = \mathbb{C} \)

Then we have, for a non-degenerate quadratic form satisfying the above condition, that

Proposition 4.10. The spin group \( \text{Spin}(Q) \) is a double cover of the special orthogonal group \( \text{SO}(Q) \). We have an exact sequence:

\[
1 \to \pm 1 \to \text{Spin}(Q) \xrightarrow{\pi} \text{SO}(Q) \to 1.
\]

Proof. Consider the element of \( \text{SO}(Q) \) generated by the product of 2k reflections: \( r_{v_1} \ldots r_{v_{2j}} \). It is in the image of \( w = v_1 \ldots v_{2j} \in \Gamma^+(V,Q) \). We have that \( N(w) = Q(v_1) \ldots Q(v_{2j}) \). If \( F = \mathbb{C} \) there exists \( a \in F^* \) such that \( a^2 N(r) = 1 \). If \( F = \mathbb{R} \) and \( Q \) is definite, \( N(r) \) is positive so there exists \( a \in F^* \) such that \( a^2 N(r) = 1 \). In either case \( av_1 \ldots v_{2j} \in \text{Spin}(Q) \). This proves that \( \pi : \text{Spin}(Q) \to \text{SO}(Q) \) is surjective.

The kernel is \( F^* \cap \text{Spin}(Q) = \pm 1 \).

Now we specialize to the case that \( F = \mathbb{R} \) and \( Q \) is the standard inner product. In this case, we will denote \( \text{Spin}(Q) \) by \( \text{Spin}(n) \). This is the universal cover of \( \text{SO}(n) \). The final ingredient is:

Proposition 4.11. The group \( \text{Spin}(n) \) is connected.

Proof. The map \( \text{Spin}(n) \to \text{SO}(n) \) is locally trivial. It is thus enough to show that \(+1\) and \(-1\) are connected in \( \text{Spin}(n) \). We have that \( \text{Spin}(n-1) \subset \text{Spin}(n) \), so it is sufficient to consider \( \text{Spin}(2) \). The corresponding Clifford algebra \( C(2) \) is the quaternion algebra \( \mathbb{H} \). We denote the standard basis of the quaternions by \( \{1, i, j, k\} \). Then \( C^+(2) = \mathbb{C} = \mathbb{R} + \mathbb{R}k \) and

\[
\text{Spin}(2) = \{ \alpha \in \mathbb{C} \ | \ |\alpha| = 1 \} = \{ \cos \phi + \sin \phi k = \exp(\phi k) \}.
\]

We have that

\[
\exp(\phi k) \bar{i} \exp(-\phi k) = \exp(2 \phi k) \bar{i},
\]

\[
\exp(\phi k) \bar{j} \exp(-\phi k) = \exp(2 \phi k) \bar{j},
\]

from which connectedness is clear.

Thus we have proven

Theorem 4.12. The group \( \text{Spin}(n) \) is the universal cover of \( \text{SO}(n) \).
References

