Vector Fields on Spheres I: Construction*

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In what follows, we’ll both produce a good deal of linearly independent vector fields on spheres, and (hopefully) satisfy the final paper requirement for Prof. Kronheimer’s Math 231br — without even a hint of spectral sequences. I lay claim to any incorrect statements or other imperfections in this presentation as my own original work.

1 Introduction

The question of vector fields on spheres is a classical one, and — if that’s not enough to whet one’s appetite — one that ends up involving some incredible mathematics. Considering the space constraints of these papers, let’s consider the problem adequately introduced, so we can move away from my rambling and towards something with content.

2 Beginnings

Everybody and his mother knows that $TS^0 \simeq \mathbb{R}^0 \oplus 0$ (a.k.a. $0$).

**Question.** How trivial is $TS^n$?

We begin the discussion with a theorem that is dear to many hearts, if only because of the name:

**Theorem 1** (Hairy Ball Theorem). Any vector field on $S^{2n}$ has a zero.

(Here, of course, $n > 0$.) There are many proofs. One goes by Poincare-Hopf, plus noting that $\chi(S^n) = 1 - (-1)^n$. So that takes care of half of the countably many questions posed above.

**Proposition 2.** There exists a nowhere vanishing vector field on $S^{2n-1}$.

Indeed, $(x_1, \ldots, x_{2n}) \mapsto (-x_2, x_1, \ldots, -x_{2n}, x_{2n-1})$ is one such, under the standard metric on $\mathbb{R}^{2n}$. Is that all? Well, of course not: else, there’d be nothing to write about. But we’ve at least taken a first step. Having produced one trivial summand, we might be foolish enough to ask:

*The “I” suggests a “II”: Presumably, after I learn (a lot) more algebraic topology, I’ll come back and write something about the corresponding upper bound, first due to Adams.*
3 Are They All Parallelizable?

— where we of course mean the odd-dimensional spheres.

For example, $S^0 \simeq \mathbb{Z}/2\mathbb{Z}$ is a Lie group (to abuse the name). So it’s no surprise that it’s parallelizable (though you’d be right to point out that 0 is even). But so is $S^1 \simeq U(1) \simeq \mathrm{SO}(2)$. Any more? Well, looking to the unit quaternions, $S^3 \simeq \mathrm{SU}(2) = \mathrm{Sp}(1)$. And, as we should expect when we see $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ together, $\mathbb{O}$ (the octonions) tags along, giving a parallelizable $S^7$. Note that this does not mean $S^7$ admits a Lie group structure — indeed, the octonions are nonassociative, so the obvious attempt doesn’t work.

So these first four examples are parallelizable — but that’s it:

Theorem 3 (Kervaire, Bott-Milnor). $S^n$ is not parallelizable for $n > 7$.

We showed this by using Bott periodicity and playing with characteristic classes on an early problem set. Great fun! But what of our original question?

4 An Observation

Let’s be a bit more systematic. Where did the construction of a nowhere zero section of $T S^{2n-1}$ come from? Well, we wrote $\mathbb{R}^{2n} \simeq \mathbb{C}^{\oplus n}$ as real vector spaces, and then we multiplied by $i$ in each slot. What of $S^3$? We multiplied by $i,j,k$ instead. Similarly for $S^7$.

We got nonvanishing from the fact that each of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ is a division algebra, and linear independence from orthogonality at the identity. (Note that this already almost shows that a finite-dimensional real division algebra must have dimension 1, 2, 4, or 8 — but not quite. We’ll see this result later.) This suggests playing a similar game for the higher-dimensional spheres. Which one of the many generalizations that come to mind to follow is not obvious, of course! (We’ll leave some of the more algebraic corollaries of this game to an appendix.)

5 Clifford Algebras and Clifford Modules

And now, finally, to the heart of the matter. Recall the time-independent Klein-Gordon equation: $(\Delta + m^2) \psi = 0$ (in natural units — i.e., $\hbar = c = 1$). Dirac was able to take the square root of the operator on the left-hand side (much like $E^2 = p^2 + m^2$ suggests $|E| = \sqrt{p^2 + m^2}$) to get the Dirac equation: $(-i \gamma^\mu \partial_\mu + m^2) \psi = 0$, where the $\gamma^\mu$ can be expressed in terms of our old friends the Pauli spin matrices, as operators on $\mathbb{C}^4$. How? Underlying the whole construction are the anticommutation relations $\frac{1}{2}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = g^{\mu\nu} \text{id}$ (OK, the $g^{\mu\nu}$ was superfluous — I only know the Minkowskian theory, anyway). That is to say, in our familiar Minkowskian case, $(\gamma^\mu)^2 = -\eta^{\mu\nu} \text{id}, [\gamma^\mu, \gamma^\nu]_+ = 0$ for $\mu \neq \nu$. If I knew any physics, I’d continue on with something about spinors, but alas...

In any case, that’s some (admittedly random) motivation for abstracting such relations. It’s worth noting that we already know examples of the above in less than 32 real dimensions — namely (essentially by definition), $1 \in \mathbb{R}$, $(1,i) \in \mathbb{C}$, and $(1,i,j,k) \in \mathbb{H}$. This won’t quite get us $\mathbb{O}$, but that’s not the point of this paper (though we will see how to get $\mathbb{O}$ in the appendix)! So, we abstract.

Definition 4. Let $V$ be a finite-dimensional real vector space, and $Q$ a quadratic form on $V$. The Clifford algebra associated to $Q$ is: $\mathcal{C}\ell(V,Q) := T(V)/\langle v \otimes v - Q(v,v) \rangle$, where $T(V)$ is the tensor algebra of $V$. 

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For example, we might take \( V := \mathbb{R}^n, Q(\cdot) := ||\cdot||^2 \), whence we’d get something we’ll note as \( \tilde{\mathcal{C}}_n := \mathcal{C}(\mathbb{R}^n, ||\cdot||^2) \). Alternatively, we might take \( V := \mathbb{R}^n, Q(\cdot) := -||\cdot||^2 \), whence we’d get something we’ll note as \( \mathcal{C}_n \). This last guy is called the Clifford algebra on \( n \) generators.

Why? Take \( e_i \), the standard orthonormal basis. Then each \( e_i \) is a square root of \(-1\), and the \( e_i, e_j \) anticommute (for \( i \neq j \)). Note also that \( \mathcal{C}_n \simeq T(\mathbb{R}^n)/\langle e_i^2 = -1, [e_i, e_j]_+ = 0 \rangle \) (since \((e_i + e_j)^2 = -2 = e_i^2 + e_j^2 \)— as usual, \( j \neq i \) throughout).

**Definition 5.** A Clifford module is a module over a Clifford algebra \( \mathcal{C}(V, Q) \).

So now we can ask: when is \( \mathbb{R}^n \) a \( \mathcal{C}_n \)-module? Why do we care?

**Proposition 6.** If \( \mathbb{R}^n \) admits a \( \mathcal{C}_n \)-module structure, then there are \( n \) linearly independent vector fields on \( S^{n-1} \).

This is just the same proof as in the case of \( S^1, S^3 \), or \( S^7 \).

**Proof.** Take the standard metric on \( \mathbb{R}^n \), and average it over the (finite!) group generated by the \( e_i \) (the result of which I’ll denote \( \langle \cdot, \cdot \rangle_{\text{avg}} \)— note that this is still symmetric). Then each \( e_i \) is an orthogonal transformation in the eyes of \( \langle \cdot, \cdot \rangle_{\text{avg}} \). Via our metric on \( \mathbb{R}^n \), identify \( T_v S^{n-1} \simeq v^\perp \subseteq \mathbb{R}^n \simeq T_v \mathbb{R}^n \). Let \( v_i : S^{n-1} \rightarrow TS^{n-1} \) via \( v \mapsto e_i \cdot v \). That these are nowhere vanishing is evident: \( e_i^2 = -1 \), so that \( e_i(e_i v) = -v \neq 0 \). That these even lie in the relevant tangent space is also evident: \( \langle v, e_i v \rangle_{\text{avg}} = (e_i v, e_i^2 v) = -(v, e_i v) \), where we have used symmetry and the fact that \( e_i \in O(N) \). Moreover, these are all orthogonal: \( \langle e_i v, e_j v \rangle = \langle e_j e_i^2 v, e_j e_i v \rangle = \langle e_j v, e_i^2 e_j v \rangle = -(e_i v, e_j v) \). So we have the claim, since evidently these are smooth. \( \square \)

Having done essentially nothing, we now look to squeeze every last drop out of this proposition. So we ask ourselves: exactly when can we apply this? That is to say,

**Question.** When does \( \mathbb{R}^n \) admit a \( \mathcal{C}_n \)-module structure?

### 6 A Classification

Let’s crystallize our earlier remarks.

**Proposition 7.**

1. \( \mathcal{C}_1 \simeq \mathbb{C} \),
2. \( \mathcal{C}_2 \simeq \mathbb{H} \).

Note: as the ugly symbol \( \cong \) suggests, these will be noncanonical. Indeed, in what follows, I will be extremely noncanonical, in order to skip through the (beautiful) theory and get to the end result. The reader will have to forgive this necessary evil.

**Proof.** Take \( e_1 \mapsto i \) for the first, \( e_1 \mapsto i, e_2 \mapsto j \) for the second. \( \square \)

We can do similar things with the “other” Clifford algebras from before:

**Proposition 8.**

1. \( \widetilde{\mathcal{C}}_1 \cong \mathbb{R} \oplus \mathbb{R} \),
2. $\widetilde{C\ell}_2 \cong \text{Mat}_2(\mathbb{R})$.

Proof. For the first, take $e_1 \mapsto (1, -1)$.

For the second, take $e_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

So we have the simplest examples down. But this is everything!

Proposition 9.

1. $\mathcal{C}\ell_{n+2} \cong \mathcal{C}\ell_n \otimes \mathbb{R} \mathcal{C}\ell_2$,

2. $\widetilde{\mathcal{C}\ell}_{n+2} \cong \mathcal{C}\ell_n \otimes \mathbb{R} \widetilde{\mathcal{C}\ell}_2$.

Proof. For the first, take e.g. $e_i \mapsto e_i \otimes e_1 e_2$

for $i \leq n$, and

$e_{i+n} \mapsto 1 \otimes e_i$

for $1 \leq i \leq 2$. Since $(e_1 e_2)^2 = -1$ in the right-hand factor, and $e_i^2 = 1$ in the left (and similarly for the second case), we get a (surjective) map. To map back, take

$e_i \otimes 1 \mapsto e_i e_{n+2} e_{n+1}$,

$1 \otimes e_i \mapsto e_{n+i}$.

Note that this does indeed factor through both quotients, and both compositions are evidently the identity.

For the second, take the same exact maps — note that $e_i, e_j$ still anticommute, since $(e_i + e_j)^2 = 2 = e_i^2 + e_j^2$ (i.e., the same proof as before). □

Immediately, we get:

Corollary 10. $\mathcal{C}\ell_{n+4} \cong \mathcal{C}\ell_n \otimes \mathbb{R} (\mathcal{C}\ell_2 \otimes \mathbb{R} \mathcal{C}\ell_2)$, and similarly for $\widetilde{\mathcal{C}\ell}_n$.

Note the hints of Bott…

Theorem 11. $\mathcal{C}\ell_{n+8} \cong \mathcal{C}\ell_n \otimes \mathbb{R} \text{Mat}_{16}(\mathbb{R})$.

Proof. From our earlier remarks,

$\mathcal{C}\ell_2 \cong \mathbb{H}, \widetilde{\mathcal{C}\ell}_2 \cong \text{Mat}_2(\mathbb{R})$,

whence

$\mathcal{C}\ell_2 \otimes \mathbb{R} \widetilde{\mathcal{C}\ell}_2 \cong \text{Mat}_2(\mathbb{H})$

(here we have implicitly used that right- and left-multiplication commute in our isomorphism). But we won’t use this last bit. Note that

$(\mathcal{C}\ell_2 \otimes \mathbb{R} \widetilde{\mathcal{C}\ell}_2)^{\otimes 2} \cong (\mathcal{C}\ell_2 \otimes \mathbb{R} \mathcal{C}\ell_2) \otimes \mathbb{R} (\widetilde{\mathcal{C}\ell}_2 \otimes \mathbb{R} \widetilde{\mathcal{C}\ell}_2)$. 

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But
\[ \text{Mat}_2(\mathbb{R}) \otimes_\mathbb{R} \text{Mat}_2(\mathbb{R}) \simeq \text{Mat}_4(\mathbb{R}), \]
and
\[ \mathbb{H} \otimes_\mathbb{R} \mathbb{H} \simeq \text{Mat}_4(\mathbb{R}) \]
via the evident maps (the last map is an isomorphism thanks to comparing dimensions and using the trace pairing on the right-hand side). Since
\[ \text{Mat}_4(\mathbb{R}) \otimes_\mathbb{R} \text{Mat}_4(\mathbb{R}) \simeq \text{Mat}_{16}(\mathbb{R}), \]
we are done. \(\square\)

Note that one can classify, in the same way, all Clifford algebras over the reals: after all, any quadratic form \(Q\) is determined by its signature — i.e., it is equivalent to one of the usual forms — and one can play the same (explicit) game for signature \((p, q)\). But, thankfully, we won’t need this here.

Tabulating our results,

**Theorem 12.**
1. \(\mathcal{C}_{\ell_1} \cong \mathbb{C} \simeq \text{Mat}_1(\mathbb{C}),\)
2. \(\mathcal{C}_{\ell_2} \cong \mathbb{H} \simeq \text{Mat}_1(\mathbb{H}),\)
3. \(\mathcal{C}_{\ell_3} \cong \mathbb{H} \oplus \mathbb{H} \simeq \text{Mat}_1(\mathbb{H}) \oplus \text{Mat}_1(\mathbb{H}),\)
4. \(\mathcal{C}_{\ell_4} \cong \text{Mat}_2(\mathbb{H}),\)
5. \(\mathcal{C}_{\ell_5} \cong \text{Mat}_4(\mathbb{C}),\)
6. \(\mathcal{C}_{\ell_6} \cong \text{Mat}_8(\mathbb{R}),\)
7. \(\mathcal{C}_{\ell_7} \cong \text{Mat}_8(\mathbb{R}) \oplus \text{Mat}_4(\mathbb{R}),\)
8. \(\mathcal{C}_{\ell_8} \cong \text{Mat}_{16}(\mathbb{R}).\)

**Proof.** The first two we’ve already seen. For the third and fourth, recall that
\[ \widetilde{\mathcal{C}}_{\ell_1} \cong \mathbb{R} \oplus \mathbb{R}, \widetilde{\mathcal{C}}_{\ell_2} \cong \text{Mat}_2(\mathbb{R}) \]
(and apply \(\mathcal{C}_{\ell_{n+2}} \cong \widetilde{\mathcal{C}}_{\ell_{n}} \otimes_\mathbb{R} \mathcal{C}_{\ell_2}\)). For the fifth, apply
\[ \mathcal{C}_{\ell_{n+4}} \cong \mathcal{C}_{\ell_n} \otimes_\mathbb{R} \text{Mat}_2(\mathbb{H}) \]
and note that
\[ \mathbb{C} \otimes_\mathbb{R} \text{Mat}_2(\mathbb{H}) \simeq \text{Mat}_4(\mathbb{C}), \]
by viewing \(\mathbb{H}\) as a two-dimensional \(\mathbb{C}\)-vector space. For the sixth, by the same token,
\[ \text{Mat}_2(\mathbb{R}) \otimes_\mathbb{R} \text{Mat}_2(\mathbb{H}) \simeq \text{Mat}_8(\mathbb{R}), \]
by viewing \(\mathbb{H}\) as a four-dimensional \(\mathbb{R}\)-vector space. This also gives 7. We’d already seen 8 above. \(\square\)

This ends the classification, since we know how to proceed from here (namely, tensor everything by \(\text{Mat}_{16}(\mathbb{R})\) to go up 8).
7 The Construction

As mentioned before, we’d like to squeeze every last vector field out of our earlier construction. So:

**Definition 13.** The Radon-Hurwitz numbers are \( \rho(N) := \max \{n|\mathbb{R}^N a \mathcal{C}_n\text{-module}\} \).

Recall that, for a division algebra \( K \), we have the following result (I had planned on presenting this, but I decided against it):

**Theorem 14.**
1. Modules over \( \text{Mat}_n(K) \) are semisimple.
2. \( K \oplus n \) is the unique simple module (up to isomorphism) of \( \text{Mat}_n(K) \).
3. The simple modules of \( \oplus_{i=1}^{N} \text{Mat}_n_i(K) \) are \( K \oplus n_i \), with exactly \( N \) isomorphism classes.

This is all developed in Lang’s *Algebra* — his 1965 edition has it, at least (and I couldn’t find any other editions in Birkhoff…).

In any case, proof or not, we therefore know exactly when \( \mathbb{R}^N \) admits a \( \mathcal{C}_n\text{-module} \) structure. Namely, let

\[
\rho'(n) := \min \{N|\mathbb{R}^N a \mathcal{C}_n\text{-module}\},
\]

a slightly different definition than before. Note that this is the same as the unique \( N \) for which \( \mathbb{R}^N \) is a simple \( \mathcal{C}_n\text{-module} \). Also, note that

\[
\rho(N) = \max \{n : \rho'(n)|N\},
\]

from semisimplicity. Further, from our periodicity result,

\[
\rho'(n + 8) = 16\rho'(n).
\]

Finally, from our tabulation,

\[
\rho'(1) = 2, \rho'(2) = 4, \rho'(3) = 4, \rho'(4) = 8, \rho'(5) = 8, \rho'(6) = 8, \rho'(7) = 8, \rho'(8) = 16.
\]

Thus, in general, \( \rho'(8a + b) = 2^{4a+1+\lfloor \log_2(b) \rfloor} \) (for \( 1 \leq b \leq 8 \)). That is to say,

\[
\rho(N) = \max\{8a + b : 2 \cdot 16^a \cdot 2^{\lfloor \log_2(b) \rfloor} | N\}.
\]

Evidently \( \rho(N) = 0 \) for \( N \) odd — this we would hope, since we’ve already seen we can’t produce any nowhere vanishing vector fields on \( S^{2n} \). For \( N \) even, write \( N = 2^{a+b}N' \), with \( N' \) odd, \( 0 \leq b \leq 3 \). Then \( \rho(N) = 8a' + b' \), where \( 4a' + 1 + \lfloor \log_2(b') \rfloor \leq 4a + b \) is maximal. That is to say, \( b' = 2^b - 1, a' = a \) (or, if \( b = 0, b' = 8, a' = a - 1 \)), which, in any case, gives

\[
\rho(N) = 8a + 2^b - 1.
\]

Hence,

**Theorem 15.** Let \( N = 2^{a+b}N' \), with \( 0 \leq b \leq 3, N' \) odd. Then: there exist \( 8a + 2^b - 1 \) linearly independent vector fields on \( S^{N-1} \).

For example, we’ve just produced 99 linearly independent vector fields on the 1125899906842623-sphere. And I had always wondered whether one could…

So we have a reasonable lower bound, that at least parallelizes the spheres that are parallelizable. In fact, this is the best we could hope for (after all, the previous example, together with our intuition for 1125899906842624-space, should suggest this): this bound is tight.
8 Maximality

The proof is, as noted before, due to Adams. It uses a great deal more than I know, but we can at least start down the path, and work out what’s going on for $S^5$ (where we’d expect, from before, exactly 1 nonzero vector field). So let’s begin. The goal is, as noted before:

**Theorem 16** (Adams). There do not exist $ρ(N) + 1$ linearly independent vector fields on the $S^{N−1}$-sphere.

Now, note that, of course, producing $n$ linearly independent vector fields is the same thing as producing $n$ orthonormal (under the standard metric) vector fields: this is Gram-Schmidt. So we are asking for a section of the fiber bundle

$$V_n(\mathbb{R}^N) \to V_{n+1}(\mathbb{R}^N) \to S^{N−1}$$

(via projection onto the last factor), where $V_k(\mathbb{R}^ℓ)$ is the relevant Stiefel manifold: the space of orthonormal $k$-frames in $\mathbb{R}^ℓ$.

**Proof.** (Opening!) Suppose we were to have such a section. Name it $s$. Equip the tautological line bundle over $\mathbb{P}^m$ with the standard metric, induced by that of $\mathbb{R}^{m+1}$. From now on, all vector bundles will be taken over $\mathbb{P}^n$.

**Lemma 1.** Such an $s$ induces a map

$$s': S(\mathbb{R}^{\oplus N}) \to S(\text{taut}^{\oplus N}),$$

where $S(\cdot)$ denotes the relevant sphere bundle. Further, $s'$ is the identity over the fiber of the basepoint $\mathbb{R}e_n$.

Note that the notation is technically superfluous:

$$S(\mathbb{R}^{\oplus N}) \simeq \mathbb{P}^{n−1} \times S^{N−1}.$$

We’ll see in a bit that it isn’t really.

**Proof.** Define

$$s': \mathbb{R}^n \times S^{N−1} \to \mathbb{R}^n \times \mathbb{R}^N$$

via

$$(v, w) \mapsto (v, s(w)v),$$

where we view $s(w) \in V_n(\mathbb{R}^N)$ as an element of $\text{Hom}_\mathbb{R}(\mathbb{R}^n, \mathbb{R}^N)$ preserving orthonormality (i.e., a matrix with orthogonal columns). Note that

$$⟨s(w)v, s(w)v⟩ = ⟨v, v⟩$$

by definition, whence

$$s': S^{n−1} \times S^{N−1} \to S^{n−1} \times S^{N−1}.$$\)

Moreover, the $\mathbb{Z}/2\mathbb{Z}$ action on the first factor of the domain commutes with the diagonal $\mathbb{Z}/2\mathbb{Z}$ action on both factors of the right-hand side. Thus, it induces a map

$$s': \mathbb{P}^{n−1} \times S^{N−1} \to S^{n−1} \times S^{N−1}/(v, w) \sim (−v, −w).$$
Notice that this last space, via projection to the first factor, forms a fiber bundle over $\mathbb{R}P^{n-1}$. To be more specific, this thing is precisely the sphere bundle of $N$ copies of the tautological bundle (e.g., $S^{n-1} \times \mathbb{R}/(v, w) \sim (-v, -w)$ is exactly the tautological bundle, since the fiber over $\ell = \{\pm v\}$ looks like $\ell = \mathbb{R}\{\pm v\}$, in the evident way). Note that $s(e_n) = v = v$ since $s$ is a section of the projection to the last factor, so we have the last claim as well.

This finishes the proof. $\Box$

To say that $s'$ is the identity over $\mathbb{R}e_n$ is to say that, over $U$ a small Euclidean open containing $e_n$ over which we have trivialized taut, $s'$ induces a map $U \to \text{Cont}(S^{n-1}, S^{n-1})$ that takes $e_n$ to the identity map. Since $U$ is path-connected, this lands in the path component of the identity map, which is to say that, over each fiber, it is homotopic to the identity. That is to say, it is a homotopy equivalence on fibers.

Now I will wave my hands:

**Theorem 17** (Dold). Let $E, E'$ fibrations over $B$, with $B$ connected and $E, E'$ homotopy equivalent to CW complexes. Let $f : E \to E'$ a bundle map such that $f$ induces a homotopy equivalence on fibers. Then: $f$ is a fiber homotopy equivalence.

That is to say, there exists a bundle map back such that the relevant compositions are homotopic to the identity through bundle maps. Since I don’t even know what a fibration is, I certainly won’t try to prove this.

Good! So we are on our way. We’ve just discovered that:

**Corollary 18.** $S(\text{taut}^{\oplus N}) \to \mathbb{R}P^{n-1}$ is fiber homotopy trivial.

(Namely, it is fiber homotopy equivalent to a trivial bundle.)

Let’s relate this to something about $\mathbb{R}P^{n-1}$ sitting inside $\mathbb{R}P^{n+N-1}$. Let $\nu := N(\mathbb{R}P^{n-1}; \mathbb{R}P^{n+N-1})$, the normal bundle of this embedding.

**Lemma 2.** $\nu \simeq \text{taut}^{\oplus n}$.

*Proof.* We’ll show that $\nu \simeq (\text{taut}^*)^{\oplus n}$. Using the metric will give us the claim.

Write $\mathbb{R}^{n+N} \simeq \mathbb{R}^n \oplus \mathbb{R}^{\oplus n}$. A linear injection $\mathbb{R} \to \mathbb{R}^{n+N}$ that doesn’t factor through the second summand exactly corresponds (via taking $\phi : \mathbb{R} \to \mathbb{R}^n \oplus \mathbb{R}^{\oplus n}$ to $(\pi_k \circ \phi)_k$, with $\pi_k$ the projection onto the $k$-th factor of $\mathbb{R}^{\oplus n}$ (thought of as a map in $\phi(\mathbb{R})^*))$ to $N$ elements of the dual of the line $\mathbb{R} \to \mathbb{R}^n$ gotten by projecting onto the first factor. That is to say, a tangent vector at $\ell$ inside $\mathbb{R}P^{n+N-1}$ can exactly be recovered by its $\mathbb{R}P^{n-1}$ part, along with $N$ maps from $\ell$ to $R$. This gives the claim. $\Box$

But the Thom space of the normal bundle is isomorphic to $\mathbb{R}P^{n+N-1}/\mathbb{R}P^{n-1}$, via

$$\mathcal{D}(\nu) \to \mathbb{R}P^{n+N-1}$$

via

$$(\mathbb{R}v, a_1, \ldots, a_N) \mapsto \left((1 - \|(a_1, \ldots, a_N)\|)v, a_1\|v\|, \ldots, a_N\|v\|)\right)!$$

(The map back on the level of quotients is the evident one.) So we have our desired translation.

Recalling that the Thom space is a homotopy invariant, this implies that

$$T(\mathbb{R}^{\oplus N}) \simeq \mathbb{R}P^{n+N-1}/\mathbb{R}P^{n-1}.$$
Alternatively, given a fiber bundle $\pi : E \to B$ with fibers $S^{n-1}$, we could construct the relevant Thom space as the mapping cone of the fiber bundle, namely $T(E) := B \cup_{\pi} E$. This would receive a canonical map from $S^n$, namely by looking over the basepoint of $B$ (which is, as always, assumed based). In this way, our old Thom space would just be the Thom space of the associated sphere bundle.

In any case, this thing is a fiber homotopy equivalence invariant (rel these canonical maps from $S^n$, even!). So now recall what was said earlier about $S(taut^{\oplus N})$ being fiber homotopy trivial. This implies that we have a map to

$$\mathbb{RP}^{n-1} \times S^{n-1}$$

which induces a homotopy equivalence on fibers. That is to say, upon projecting to $S^{n-1}$, we have a bundle map

$$S(taut^{\oplus N}) \to S^{n-1}$$

over $\mathbb{RP}^{n-1} \to pt$ that is a homotopy equivalence on fibers. Taking Thom spaces of this, we have a map $T(taut^{\oplus N}) \to S^n$ such that precomposing with the canonical map from $S^n$ induces a homotopy equivalence. Or, in other words, we have reduced to the problem of figuring out when there is an extension of the evident map $\mathbb{RP}^n/\mathbb{RP}^{n-1} \simeq S^n$ to a map $\mathbb{RP}^{n,N-1}/\mathbb{RP}^{n-1} \to S^n$.

This is the “coreducibility” mentioned in Adams’ paper. Pardon the QED symbol to follow: nothing has been demonstrated here.

Apparently the fact that the Thom space (since it is equivalent to the sphere bundle of a once-stabilized bundle, upon crushing a particular section to a point) and its canonical map from $S^n$ appears is supposed to suggest that $K$- and $J$-theory should play a role, but I certainly don’t know anything about $J$-theory, so here will end the discussion.

9 An Example

Take $N = 6$. Then, the claim is that we can’t find two trivial summands inside $TS^5$. Indeed, if we could, $TS^5$ would be isomorphic to a twice-stabilized bundle. Thankfully, we know exactly which spheres are parallelizable, and $S^5$ isn’t on the list. So it suffices to show such a thing would necessarily be trivial.

What is stabilization? Focusing on clutching functions, it is just the application of the map

$$\pi_k(SO(\ell)) \to \pi_k(SO(\ell + 1)),$$

for suitable $k, \ell$. But we’ve seen these guys before! (I am referring to the relevant fibration, which we won’t actually use.) Our claim, translated thus, is that

$$\pi_4(SO(3)) \to \pi_4(SO(4)) \to \pi_4(SO(5))$$

is the zero map. Note that the first group is

$$\pi_4(SU(2)) \simeq \pi_4(S^3) \simeq \mathbb{Z}/2\mathbb{Z}.$$ 

The second group is

$$\pi_4(S^3 \times S^3) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$
(since $S^3 \times S^3$ is the universal cover of SO(4)). Note that the map $S^3 \to SO(3)$ is $z \mapsto (z) = \ell_z$, left-translation by $z$. The map $S^3 \times S^3 \to SO(4)$ is $(z, w) \mapsto \ell_z \circ r_w$, left- and right-translations. Further, the stabilization map, if we stabilize by taking SO(3) to fix $1 = 1 + 0i + 0j + 0k$, is just $z \mapsto (z, z^{-1})$.

So our map $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is the diagonal map! (— via Eckmann-Hilton and the fact that $-1 = 1$ in $\mathbb{Z}/2\mathbb{Z}$.) Now, consider the effect of the switching map $S^3 \times S^3 \simeq S^3 \times S^3$ on the level of homotopy groups. The point is that $(\ell_w \circ r_z)\alpha = w\alpha z^{-1} = (z\alpha^{-1}w^{-1})^{-1} = (-1) \circ (\ell_z \circ r_w) \circ (-1)\alpha$ — i.e., the switching map amounts to conjugating by the inversion map (which, since we are on the unit sphere, is the same as $\alpha \mapsto \bar{\alpha}$), which does nothing to the $\mathbb{Z}/2\mathbb{Z}$’s, since it is an involution. That is to say, switching does not affect the map. Another way to say this is that the map does not factor through the projection onto either summand. It thus sends the diagonal, which certainly consists of 2-torsion elements, to 0. But this was the claim. (By the way, note that $\pi_4(S^5) = 0$ tells us that $\pi_4(SO(5)) \simeq \mathbb{Z}/2\mathbb{Z}$, which wouldn’t have helped us anyway, I think.)

10 Appendix: Algebraic Corollaries of Our Work, and the Cayley-Dickson Construction

The fruits of our labor are plenty. For instance:

**Theorem 19.** Let $K$ be a finite-dimensional division algebra over $\mathbb{R}$. Then: $\dim K = 1, 2, 4, \text{ or } 8$.

**Proof.** Such a thing parallelizes $S^{\dim K - 1}$.

OK, so that wasn’t quite a corollary of our work here. The true corollary would be as follows:

**Theorem 20.** Let $K$ be a finite-dimensional normed division algebra over $\mathbb{R}$, of dimension $n$. Then: $\mathbb{R}^n$ admits the structure of a $\mathcal{C}_l_{n-1}$-module.

The proof is rather nice, but, to be honest, significantly more work than would be reasonable here, since we’ve already proved something stronger.

Here’s another great corollary. Let’s call an $n$-square identity a collection $c_1, \ldots, c_n : \mathbb{R}^n \otimes \mathbb{R}^n \to \mathbb{R}$ such that

$$\left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i \right)^2 = \sum_{i=1}^n c_i(a_1, \ldots, a_n, b_1, \ldots, b_n)^2.$$

**Theorem 21.** Suppose there exists an $n$-square identity. Then: $n = 1, 2, 4, \text{ or } 8$.

For instance, there does not exist a 3-square identity, since $63 = (1^2 + 1^2 + 1^2)(1^2 + 2^2 + 4^2)$, but $63 \equiv 7 \pmod{8}$, which can’t be reached by a sum of only three squares (an example of Legendre, cribbed shamelessly from [4]).
Proof. Equip $\mathbb{R}^n$ with the following $\mathbb{R}$-algebra structure:

$$(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) := (c_1, \ldots, c_n).$$

Note that this is norm-preserving, by definition, whence it has no zero-divisors. But there is no (evident) unit! So we will force one: choose $u \in S^{n-1}$. Observe that, since our $\mathbb{R}$-algebra has no zero divisors, multiplication on the left or right by $u$ is injective, hence an isomorphism. Let

$$a \ast b := (r_u^{-1} a) \cdot (\ell_u^{-1} b).$$

This also gives a normed $\mathbb{R}$-algebra structure to $\mathbb{R}^n$, with no zero divisors, as before. But here $u \cdot u$ is a unit, whence we have a finite-dimensional normed division algebra over $\mathbb{R}$. Thus, the result. 

So much for sums of squares.

Something that has always been quite confusing to me is the confluence of miracles behind the existence of the cross product. Massey tells us that, besides a 7-dimensional analogue, such a thing really is special to 3 dimensions:

**Theorem 22** (Massey). Let $n \geq 3$. Suppose there is a map

$$\cdot \times \cdot : \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{R}^n \rightarrow \mathbb{R}^n$$

such that, for each $u, v \in \mathbb{R}^n$ (and under the standard inner product),

$$u \times v \perp u, v,$$

and

$$||u \times v||^2 + \langle u, v \rangle^2 = ||u||^2||v||^2.$$

Then: $n = 3$ or 7.

**Proof.** Equip $\mathbb{R} \oplus \mathbb{R}^n$ with the structure of an $\mathbb{R}$-algebra via

$$(a, v) \cdot (b, w) := (ab - \langle v, w \rangle, aw + bv + v \times w).$$

Note that the norm of the product is

$$a^2b^2 + ||v||^2||w||^2 + a^2||w||^2 + b^2||v||^2 = (a^2 + ||v||^2)(b^2 + ||w||^2),$$

after applying the two properties. Hence, the structure has no zero divisors, and so, by the same trick as before, we get a finite-dimensional normed division algebra over $\mathbb{R}$ of dimension $n + 1$. This gives the claim. 

Finally, I’d be remiss if I were to gloss over the Cayley-Dickson construction. Recall that the Clifford algebras we defined before gave us $\mathbb{R}, \mathbb{C}$, and $\mathbb{H}$, but not $\mathbb{O}$ — indeed, $\mathbb{O}$’s lack of associativity precluded such a possibility. So how might we get all four in one shot? Let $A$ be an algebra, with involution $a \mapsto a^*$ (e.g., $\mathbb{C}$, with complex conjugation, or $\mathbb{R}$, with the identity map). Define an algebra structure on $A \oplus A$ as follows:

$$(a, b) \cdot (c, d) := (ac - d^*b, da + b^*c).$$

Define an involution on $A \oplus A$ via

$$(a, b)^* := (a^*, -b).$$
This certainly gives $\mathbb{C}$ from $\mathbb{R}$, and, after some thought, certainly $\mathbb{H}$ from $\mathbb{C}$. In fact, it gives $\mathcal{O}$ from $\mathbb{H}$! Moreover, properties of the construction explain why we are “losing a property” at each step, and so why the construction applied to $\mathcal{O}$ does not give a nice object (in fact, it gives the so-called sedenions, which have zero divisors).

At any rate, so ends our discussion.

References


