CDS Spreads and Systemic Financial Risk∗

Stefano Giglio
Harvard University

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Abstract

This paper investigates the information content of bond and Credit Default Swap prices of financial institutions for the measurement of systemic risk in the financial sector, defined as the probability that several institutions default. Because CDS contracts involve counterparty risk, this is reflected in their price, the spread paid to a dealer to insure against default. Then, the set of spreads of all CDSs written by each member of the financial network on the other members - together with bond prices - reflects the risk-neutral probabilities of default of each institution and of each pair of institutions in the network. In the paper, I show how this information can be aggregated to construct bounds on the probability of systemic events. These bounds are proven to be the tightest ones possible given this information set. Applying this method to the recent financial crisis yields allows to decompose movements in CDS spreads and bond yields in a systemic and an idiosyncratic default components. I show that the spikes observed around March and September 2008 are attributable to idiosyncratic risk, not systemic risk. Systemic risk was low until mid 2008 and increased steadily after then. Essential to this result is the use of both bond and CDS spreads: alternative measures that only look at bonds or CDSs fail to capture this decomposition, and report sharp increases in systemic risk already in 2007.

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1 Introduction

The risk of simultaneous collapse of several large financial institutions (LFIs) - once considered the most solid firms on Wall Street - has risen to prominence during the recent financial crisis. Both for regulators and for private agents that interact with financial markets, assessing the risk of multiple defaults of financial institutions is a crucial step in order to take optimal decisions. Not only it is important to understand how likely are shocks that can cause the default of several LFIs; it is also fundamental to distinguish the relative probabilities of default events of different severity.

This paper deals with the measurement of systemic financial risk, defined as the probability of joint default of several large financial institutions. In particular, it focuses on systemic default risk of degree $r$, the probability that at least $r$ financial institutions collapse in a short time horizon. Measuring such risk is a difficult task. On the one hand, measures constructed looking directly at the books of financial institutions are backward-looking in nature and limited by the complexity of the positions and risks involved as well as by the availability of data. On the other hand, measures that try to learn about systemic risks by looking at the historical distribution of the returns on the assets of LFIs are generally unreliable because they involve estimating joint tail probabilities from limited time series, which is possible only under strong parametric assumptions$^1$.

The most widely used measures of systemic risk try to circumvent these limitations by estimating default probabilities of individual financial institutions implied in the prices of traded securities such as equity options or Credit Default Swaps (CDSs)$^2$, and then aggregating this information to learn about systemic risks. These measures are forward-looking and reflect the information set of market participants. Figure 1 plots two simple examples of such market-based measures of default risk in the financial sector: the average yield spread and the average CDS spread of the largest 14 financial institutions$^3$. The yield spread (the yield on a firm’s bonds in excess of the risk-free rate) and the CDS spread (the cost of insuring against the firm’s default) both reflect the probability that a firm defaults. The idea behind these measures is that an increase in systemic risk in the financial sector should cause the risk of default of each institution to increase, and therefore lead to an increase in both the average yield spread and the average CDS spread. The account of the crisis these measures give is of an increase in systemic risk starting in August 2007, followed by several episodes in which systemic risk spiked (such as around March 2008, September 2008 and then March-April 2009), and a final drop starting in 2009.

Unfortunately, the existing market-based measures can be misleading for two reasons. First, they involve strong modeling assumptions in aggregating the individual risks of financial intermediaries into estimates of systemic risk. For example, the CDS-based measure reported in Figure 1, which shows the average CDS spreads of a group of dealers, is only informative about the joint distribution of defaults if

$^1$For a more detailed discussion of the literature, see the end of this Introduction.
$^2$A Credit Default Swap is an insurance contract against the default of a firm, for example a financial institution. See Section 2 for details on the contract.
$^3$The Figure replicates the Counterparty Risk Index, produced by Credit Derivative Research. These institutions alone constitute more than 90% of the CDS protection sold.
we make assumptions about the relationship between marginal and joint default probabilities. Second, measures based on the prices of securities traded over the counter (OTC) ignore counterparty risk. As I will show below, ignoring counterparty risk introduces a bias that increases precisely when the financial system is distressed.

In this paper I propose a novel measure of systemic risk, based on a combination of bond prices and CDS spreads, that exploits the pricing of counterparty risk in CDS contracts. The measure improves over other market-based measures, by optimally aggregating the information contained in security prices into bounds on systemic risk - the probability of default of several institutions. These bounds are robust to modeling assumptions on the relationship between risks of individual institutions and joint risks. Unlike the alternatives, this measure allows a decomposition of bond yields and CDS spreads into an idiosyncratic component and a systemic component. In addition, it allows to track over time the contribution to systemic risk of each institution.

The starting point for the analysis is the pricing of counterparty risk in CDS contracts. The price of insurance (CDS spread) written by a financial institution against the default of a bond reflects both the probability of default of the bond issuer, called reference entity, and the correlation between its default and the default of the protection seller. In particular, the value of the insurance decreases as this correlation increases. The price of the bond, instead, reflects only the marginal probability of default of the firm which issued the bond\(^4\). Combining the price of the bond of a company with the spread of the CDS written on that company by a particular financial intermediary, we can learn about the joint default risk of that company and the protection seller.

A standard way to see this is to look at the bond/CDS basis. By buying the bond and insuring it with the corresponding CDS, one obtains a risk-free debt security as long as there is no counterparty risk in the CDS contract. An approximate arbitrage relation then says that, in the absence of counterparty risk, the yield spread on the bond should be equal to the corresponding CDS spread: the bond/CDS basis (the difference between the two) should be zero. Counterparty risk, by lowering the CDS spread without affecting the yield spread of the bond, produces a nonzero basis\(^5\). The bond/CDS basis therefore will contain information about the joint default risk of the reference entity and the protection seller. Figure 1 shows that on average the bond/CDS basis for financial institutions is generally quite different from zero. In this paper, I use implicitly the information about joint default risk contained in the basis of different financial intermediaries.

If we consider the CDS protection sold by each financial institution against the default of every other institution, and we combine it with the corresponding bond yield spreads, we can in principle learn the joint default risk of any two financial intermediaries, as well as the marginal default probability of each institution. While this information set might at first seem rich enough to completely pin down systemic risk in the financial network, this is not the case. Usually, modeling assumptions are made to obtain a point

\(^4\)Of course, other factors affect bond prices as well. The most important one, liquidity, is explicitly considered later.

\(^5\)If \(y - r^F\) is the yield spread of a bond over the corresponding risk-free security and \(z\) is the CDS spread, the basis is usually quoted as \(z - (y - r^F)\). Counterparty risk therefore introduces a negative basis.
estimate of the joint default probability of several banks (systemic risk) given the available information set. In this paper, instead, I use the more robust approach of obtaining optimal bounds consistent with the observed prices. In particular, I construct the tightest bounds on the average monthly probability of joint default of at least \( r \geq 1 \) LFIs (for different values of \( r \)), for the period spanning January 2004 to June 2010.

Three limitations affect the construction of the bounds. First, the presence of an unobserved liquidity process in the bond market confounds the filtering of individual default probabilities out of CDS spreads, and therefore complicates the estimation of systemic risk. However, mild assumptions on the liquidity process allow a rather precise measurement of systemic default risk. In particular, I show how to obtain optimal bounds when only a lower bound is imposed on the liquidity process. I show how this lower bound can be calibrated from the time-series of each bank’s bond and CDS spreads, as well as using the cross-section of non-financial firms of similar credit rating.

Second, for every reference entity, I observe only an average of the quotes posted by the main counterparties, so my information set is smaller than the ideal one\(^6\). However, because they combine information from bond prices as well, the bounds I construct are still informative about asymmetries in the distribution of risks across the network.

Third, this paper obtains risk-neutral, not objective, systemic default probabilities. These probabilities are interesting per se because they reveal the perception of the markets about the severity of these states of the world. In addition, it is reasonable to assume that they can be considered upper bounds on the objective default probabilities, and so the upper bound obtained with this methodology is also an upper bound on systemic risk under objective probabilities. Finally, I show that the paper has additional strong implications for objective probabilities as well. However, it is important to keep in mind that in general the bounds capture variation in risk premia as well as variation in objective default probabilities\(^7\).

What emerges from this analysis is that before mid 2008 very high systemic risk in financial markets, especially around the times when CDS spreads of these banks spiked, would not be consistent with the information contained in bond and CDS prices. Throughout the crisis the level of systemic risk has been lower than other measures indicate, particularly so for the key episodes of Bear Stearns and Lehman Brothers (i.e. March and September 2008). While the optimal bounds are a complex function of the constraints imposed by the set of observed prices, the intuition for the result is straightforward. If perceived systemic risk had increased in March and September 2008, the price of insurance on large dealers - purchased from other large financial institutions - should have dropped considerably. But this did not happen. The relatively high equilibrium spreads during this period impose a tight upper bound on the amount of systemic risk perceived in financial markets. Equivalently, the bond/CDS basis - which reflects joint default risks - did not increase during 2007 and 2008 in the way the yield spread and the CDS spreads

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\(^6\)This limitation is also the reason I look at the group of dealers who are counterparties to most contracts (more than 90% of the market). This way, I can be confident I am including most of the dealers from whom the CDS quotes are obtained.

\(^7\)Anderson (2009) makes this point comparing risk-neutral default processes obtained from CDS spreads with objective processes obtained using historical data on defaults.
did, as Figure 1 shows in an aggregate form. The difference in the time series between these two series and the basis (the difference between them) is very informative about systemic risk. This intuition is developed formally in the paper, where all the information available is used to study systemic risk (and not only the average bond/CDS basis).

In addition, the bounds show that in March and September 2008, idiosyncratic default risks of some institutions, as well as joint default risk of a few banks, were extremely high. During these periods, the bounds on the probability of at least one institution defaulting spike, and get much tighter. The probabilities that several banks fail (at least 2, at least 3, and so on) instead are low, and again very tight. This methodology then allows, for these periods, a stark decomposition between idiosyncratic and systemic default risk. As discussed in the paper, this decomposition is obtained for risk-neutral probabilities but holds even more strongly for objective probabilities.

Finally, the methodology presented in this paper allows to study the configuration of the network over time, as well as the contribution to systemic risk of each institution, calculated at the upper bound of systemic risk. This analysis shows that markets anticipated by more than a month a sharp increase in the joint default probability of two key institutions, Lehman Brothers and Merrill Lynch, which in fact ended up in trouble in the same weekend (13-14 September 2008). Similarly, the upper bound attributes to these two banks a high contribution to systemic risk (the probability of a systemic event in which they would be involved).

The paper proceeds as follows. After a brief literature review, Section 2 presents an introduction to Credit Default Swaps and counterparty risk. Section 3 presents the theory of the optimal probability bounds, and Section 4 discusses the main issues of implementation. Section 5 describes the data, and Section 6 presents the results. In Section 7, I run a series of robustness tests. Section 8 concludes.

1.1 Related literature on measuring systemic risk

The literature on measures of systemic risk in the financial sector is large and has seen new impulse following the financial crisis of 2007-2009. Four categories of papers can be identified, based on different methods they use to quantify systemic risk and to study the relative contribution of each bank to this risk. A first, structural, approach looks directly at the books of financial institutions in order to learn about the distribution of joint shocks. A second approach looks at the historical joint distribution of returns and uses it to make inference about the probability of joint tail risks. A third approach uses the forward-looking information contained in securities prices (particularly derivatives) to learn directly about the implied tail risks. Finally, contrary to the other methods, a fourth approach consists in finding variables that correlate or predict systemic events, without focusing explicitly on the joint risk distribution.

The first approach is followed by Lehar (2005). In his paper, he uses the Merton (1974) model to obtain the time-series of the market value of banks’ assets, using observed equity prices and balance sheet information. By computing the correlations between the asset returns, he is able to produce measures
of joint default risk (as well as expected losses in case of default) of groups of institutions, under the assumption of multivariate normality of asset returns. Gray, Merton and Bodie (2008) propose using a similar approach to measure systemic risk not only within the financial sector but also across sectors and countries.

The structural approach requires strong assumptions about the liability structure of financial institutions, as well as about the marginal and joint distribution of risks. To overcome these difficulties, a second, reduced-form approach looks at the historical distribution of returns trying to estimate the joint default risks. For example, Acharya, Pedersen, Philippon and Richardson (2010) compute a measure of individual contribution to systemic risk by studying the average equity returns during periods of negative returns for the financial sector as a whole: a bank is deemed to have a higher contribution to systemic risk if particularly bad returns are observed when the financial sector is doing badly. Similarly, Adrian and Brunnermeier (2009) use quantile regression to estimate the VaR of the financial sector as a whole conditional on each individual banking experiencing a VaR loss in its asset values.

The reduced-form approach that looks at historical returns (or implied asset returns) has the disadvantage of trying to learn about tail events from limited time series of returns. In other words, it must infer about rare joint tail risks looking at relatively frequent losses in individual and aggregate asset values. This requires assumptions about the tail behavior of marginal and joint return distributions. A third branch of the literature tries to avoid this problem by looking directly at the probabilities of tail risks implied by derivatives, whose price is very sensitive to these precise risks. Because no traded security reflects directly the joint default risk of the largest financial institutions, papers in this category typically extract marginal default risk information from CDS spreads, and make inference about joint default risk by aggregating them using a certain copula. Huang, Zhou and Zhu (2009) use the Normal and Student-t copula and estimate covariances using intra-day equity returns. Avesani, Pascual and Li (2006) assume a latent factor structure for equity returns and relate the joint default risk of LFIs to the joint distribution of the factors. They also assume a multivariate Normal distribution. Segoviano and Goodhart (2009) instead aggregate the marginal default information obtained from CDS spreads using the CIMDO copula (Segoviano (2008)). Among these papers, those which apply the copula-based approach to study the financial crisis (as do Huang, Zhou and Zhu (2009) and Segoviano and Goodhart (2009) ) find a time-series of systemic risk that closely resembles the simple measures plotted in Figure 1, with large spikes around March and September 2008 and a steep increase in systemic risk starting in August 2007. These results are in sharp contrast with those presented in this paper.

This paper, which belongs to the third category, starts from the observation that even without assuming a copula for the joint distribution of asset values we can learn about the joint distribution of defaults, by constructing bounds. Besides, it uses the information contained in the bond/CDS basis to learn directly about joint tail risks.

Several other papers have proposed measures of systemic risk that do not result in estimates of joint default risks. For example, Kritzman, Li, Page and Rigobon (2010) propose using the fraction of the variance explained by the first principal components of CDS spreads of large financial institutions as a
measure of systemic risk in the financial sector. Other papers, instead, try to use individual or aggregated financial indicators to empirically predict financial crises, often in a cross-country setting. Examples of this approach are Poghoshan and Cihak (2009), Cihak and Schaeck (2007), Demirguc and Detragiache (1998 and 1999), and Gonzales-Hermosillo (1999).

2 Credit Default Swaps and Counterparty Risk

This section discusses the sources of counterparty risk in CDS contracts. I first describe the main characteristics of CDS contracts and the evolution of the CDS market in the recent years. Then, I discuss why counterparty risk in these contracts arises mainly from the possibility of double default of the reference entity and the counterparty, an event which revealed itself prominently during the crisis. This risk can be sizable if defaults of financial institutions are not independent. Finally, I argue that collateral provisions, when present, are unlikely to eliminate this risk - and therefore, that counterparty risk should be priced in CDS spreads.

2.1 The Credit Default Swaps Market

Credit Default Swaps are credit derivatives that allow the transfer of the credit risk of a firm between two agents. In a plain vanilla single-name CDS contract, the protection seller offers the protection buyer insurance against the default of an underlying bond issued by a certain company (called the reference entity). In the event of default by the reference entity, the seller commits to buy the bond for a price equal to its face value from the protection buyer. In exchange for the insurance, the buyer pays a quarterly premium, called CDS spread, quoted as a percentage of the notional value insured. If default occurs, the contract terminates, and the quarterly payments are interrupted. If default does not occur during the life of the contract, the contract terminates at its maturity date.

While in general these contracts are traded over the counter and can be customized by the buyer and the seller, the contract with maturity of 5 years is relatively standardized. Its market is very liquid, in terms of low transaction costs to initiate a contract with a market maker on short notice, and quotes for this contracts are regularly obtained by financial data companies from the main dealers (see Blanco et al. (2003) and Longstaff et al. (2005)).

The CDS market has grown very quickly in the last few years. Notional exposures grew from about $5tr in 2004 to around $60tr at its peak in 2007, and despite the financial crisis the total exposure is still around $40tr. The main reason for this growth in gross terms is that, due to the high liquidity of the CDS market, the easiest way to adjust the exposure to credit risk has been to enter new CDS contracts.

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8In practice, the terms of the CDS could involve physical delivery of the defaulted bond or cash settlement. In the former case, usually any bond of equal seniority can be delivered (for example, for the CDS written on a senior unsecured bond, any other senior unsecured bond of the firm could be delivered). In addition, the credit event could include restructuring and downgrading of the reference bond. These clauses have a potential effect on the price of the CDS, discussed in Section 4.
(possibly offsetting the existing ones), rather than operating directly in the bond market or cancelling CDS agreements already in place. At the center of this network of CDS contracts, a few main dealers operated with very high gross and low net exposures, emerging as the main counterparties in the market. For example, Fitch Ratings (2006) states that in 2006 the top 10 counterparties (all broker/dealers) accounted for about 89% of the total protection sold. In addition, these dealers figure in the list of top reference entities for CDS contracts; they are also the firms on which most credit protection was sold. The picture that emerges is that of a very interconnected network composed of a few large intermediaries, linked to each other both by counterparty exposure and direct credit exposure. With the crisis, the market concentrated even more, after the disappearance of some of its key players (Fitch (2008)).

2.2 Counterparty Risk

Traded over the counter, a CDS contract involves counterparty risk: the protection seller might default during the life of the CDS and therefore might not be able to comply with the commitments implied by the contract. In this case, the holders of CDS claims would still recover part of the expected payment they are due under the contract. Like other derivatives, CDS claims are treated pari passu with senior unsecured bonds, but in addition they are protected by “safe harbor” provisions, which exempt them from automatic stay of the assets of the firms, so that they can immediately seize any collateral that has been posted for them. In addition, positions across different derivatives with the counterparty can be netted against each other. The latter potentially increases the recovery in case of counterparty default, but only if the buyer finds herself with large enough out-of-the-money positions with the seller when he defaults.

In the case of early termination of the contract due to seller default, the seller usually has to compensate the buyer for the replacement cost of the contract, i.e. the cost of initiating a new insurance contract with another protection seller. This claim is small as long as the default risk of the reference entity does not jump substantially, relative to the original terms of the contract. The larger the change in CDS spread of the reference entity when the seller defaults, the larger the claim of the buyer against the defaulted counterparty. In the extreme case, if the default of the seller occurs simultaneously with the default of the reference entity, the payment due under the contract would be equal to the full insurance payment. If the default happens in a sudden way, and the buyer does not have time to call for more collateral from the counterparty, the loss incurred by the buyer may be substantial.

A stylized two-period example of the pricing of bonds and CDSs can be useful to see the role of counterparty risk in a simple way. We will consider two dealers, banks 1 and 2, which have issued a zero-coupon bond with a face value of $1 maturing at time 1, and the CDS contract written at time 0 by each of them against the default of the other one. Call $A_i$ the event of default of institution $i \in \{1, 2\}$ at time 1. Call $P(A_i)$ the probability of default of bank $i$, and $P(A_1 \cap A_2)$ the probability of joint default at time 1. All probabilities are risk-neutral. Call $R$ the (certain) recovery rate on the bond in case of default.

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9The role of counterparty risk in CDS spreads has been studied by Hull and White (2001b), Jarrow and Yu (2001), and more recently, in a context of rare disaster risk, by Barro (2010). To reduce counterparty risk, there are now several proposal to create a centralized clearinghouse to reduce counterparty risk. For a detailed discussion, see Duffie and Zhu (2010).
and suppose for now that in the event of joint default the CDS claim recovers $S \geq R$. Finally, assume that the risk-free rate between periods 0 and 1 is 0.

In this setting, the price of the bond issued by $i$, $p_i$, is determined as:

$$p_i = (1 - P(A_i)) + P(A_i)R = 1 - P(A_i)(1 - R)$$

If there is no counterparty risk in the CDS contract, the insurance premium $z_i$, or CDS spread, paid at time 0 to insure that bond is:

$$z_i = P(A_i)(1 - R)$$

It is easy to see that between the bond and the CDS there is a theoretical arbitrage relation (which in fact holds beyond this simple example, see Longstaff et al. (2005)). Consider now the case in which there is counterparty risk in the CDS contract. Then, the spread paid to buy insurance from $j$ against $i$'s default will be:

$$z_{ji} = [P(A_i) - P(A_1 \cap A_2)](1 - R) + P(A_1 \cap A_2)(1 - R)S$$

$$= [P(A_i) - (1 - S)P(A_1 \cap A_2)](1 - R)$$

which decreases with the probability of both institutions defaulting $P(A_1 \cap A_2)$\(^{10}\). The arbitrage relation with the bond is broken.

In this simple example, I have considered only two cases of counterparty risk: the case of simultaneous default of the seller and the reference entity, with the corresponding loss of $(1 - R)(1 - S)$ to the protection buyer, and the case of default by the seller alone, with no loss to the protection buyer. In reality, it is possible that the defaults of the seller and the reference entity do not occur simultaneously, yet the buyer of protection incurs losses of the same order of magnitude as if they did. This can happen, for example, if the seller’s default triggers a jump in the default probability of the reference entity, which might end up defaulting only some time later. In this case, the contract would be highly in the money immediately after the seller’s default, and could stay at those levels until the default of the reference entity. Similarly, the buyer might suffer a loss if the default of the reference entity triggers the default of the counterparty (even if default occurs some time later, but before the buyer can collect the amount due): for example, because the seller did not adequately hedge the credit risk of the reference entity. In all these cases, the two defaults do not happen simultaneously, but they are connected in such a way that the protection buyer still suffers a potentially large loss on her claim. I refer to all these cases as double default cases, even if the two defaults do not happen in the same day.

Of course, it is also possible that the value of the contract increases when the seller defaults, but the reference entity does not default for a while, or does not default at all. In some cases, this might still

\(^{10}\)It is important to realize that the order of magnitude of counterparty risk could in theory be as high as the spread itself. While in models where defaults are independent we have $P(A_1 \cap A_2) = P(A_1)P(A_2)$, most observers of the crisis would agree that defaults of major dealers are far from independent, and therefore the probability of the joint default can be of a much larger order of magnitude.
induce a loss to the protection buyer. Given collateralization, and conditional on the reference entity surviving for enough time, this case is usually not considered a major source of counterparty risk. As long as the default risk of the reference entity remains of the same order of magnitude, the replacement cost will be sufficiently small that the collateral posted will allow the recovery of most of it. The cases in which the reference entity survives the seller are more likely to be cases in which the jump in the value of the contract is smaller.

Taking into account these cases, I model CDS spreads as follows. I discretize the contract considering a time horizon short enough (a month) for which I can assume that whenever two LFIs default in the same period, their defaults are connected, and constitute a double default case. Then, the buyer is entitled to receive $(1 - R)$ from the counterparty but only recovers a fraction $S$. If instead the seller defaults in a month but the reference entity survives until future periods, I assume that the cost of replacing the contract is small enough that collateral posted can cover it: I assume a 100% recovery rate on the replacement cost in these cases. Section 4 discussed more in detail the pricing assumptions for CDSs.

### 2.3 Collateral Agreements and Pricing of Counterparty Risk

In order to protect the buyers against counterparty risk, some contracts involve a collateral agreement, under which collateral calls are tied mechanically to changes in the value of the CDS contract, as well as to downgrades of the rating of the protection seller. While helpful in reducing counterparty exposure, standard collateral agreements could hardly eliminate the counterparty risk coming from double default, for several reasons.

First, according to the ISDA Margin Survey 2008, only about 66% of the nominal exposure in credit derivatives (of which CDSs are the most important type) had a collateral agreement at all in 2007 and 2008; this number was even lower in the years before. Besides, the same document shows that collateral agreements were employed much less frequently when the counterparty was a LFI.

Second, the extent to which collateral covers against counterparty risk is limited by the jump properties of the default events - not only of one institution at a time, but of several ones within short horizons of time. As long as defaults are relatively anticipated, slow adjustment of collateral posted can basically remove all counterparty risk. However, especially for financial intermediaries, defaults often occur in very short horizons, so that the buyer does not have the possibility of obtaining enough collateral to cover all the losses in time. The Lehman bankruptcy is a clear example of this. Until the weekend of September 15th, in which Lehman collapsed and two other large financial institutions needed to be bailed out (Merrill Lynch and AIG), the default risk of these LFIs was deemed low, as reflected by low CDS spreads and high credit ratings. The joint collapse of the three institutions happened in a sudden way, so that collateral adjustment would have been impossible. As it turned out, a double default event did not materialize, because of the government bailout - therefore, buyers of Merrill and AIG CDSs from Lehman did not experience large losses. However, these events show that the risk of simultaneous collapse of several banks was relevant, and that standard collateralization practices would not have prevented large losses to buyers.
of CDS contracts, had the government decided not to intervene.

Finally, note that the collateral calls themselves, if large enough, can cause the default of the protection seller. This problem happened for example to AIG in September 2008. The collateral channel can in fact create an additional channel for joint default of the seller and the reference entity, and therefore increase the possibility of double default.

Therefore, the presence of collateral agreements improves but does not solve the problem of counterparty risk related to double default. Since this risk will be present in the final contracts, it should be priced in by rational agents, which will require higher compensation (in form of a lower CDS spread) whenever the risk of double default increases.

Appendix A discusses more in detail the evidence for the insufficiency of collateral to cover against double default cases, as well as the pricing of this risk into the CDS spreads. It also discusses the role of other factors that can be reflected in the CDS spread beyond counterparty risk.

3 Probability bounds: theory

This section develops the theory of the probability bounds on systemic default events for a network of $N$ institutions in which bond prices and CDS spreads are observed. I start with an introductory example that explains the main ideas. Then, I show how to use linear programming theory to solve the general problem, and I derive some properties of the optimal bounds. In this section, I assume that bond prices perfectly reflect the default risk of the issuer. I leave for Section 4 a detailed discussion of the implementation of the bounds, including the adjustments needed to take into account the liquidity component present in bond prices.

3.1 Probability bounds on systemic risk: an introductory example

Suppose that the financial sector consists of only three intermediaries - banks 1, 2 and 3. Since they are the only intermediaries in the market, protection against the default of $i \in I \equiv \{1, 2, 3\}$ must be bought from a $j \in I \setminus i$, i.e. one of the other two intermediaries. Following the simple pricing model presented above, we can write the bond prices ($p_i$) and the CDS spreads sold by bank $j$ on the bond of $i$ ($z_{ji}$) as:

$$p_i = 1 - (1 - R) \cdot P(A_i)$$

$$z_{ji} = (1 - R) \cdot (P(A_i) - (1 - S)P(A_i \cap A_j))$$

where $R$ is the (known) recovery rate on bonds and CDS and all probabilities are risk-neutral.

Note first that if we observe all bond prices $p_i$ and all CDS spreads $z_{ji}$, we can learn the implied marginal and pairwise risk-neutral probabilities of default. Bond prices reflect directly the marginal probability of default of each institution, while CDS spreads allow to recover pairwise default probabilities after filtering.
out the marginal probabilities. Because this information set contains the default probabilities of one or at most two institutions, but contains no direct information on the probability of the default event of all three institutions, call this a probability information set of order two. For example, suppose we extract from the prices
\[ P(A_i) = 0.2 \quad \forall i \]
\[ P(A_1 \cap A_2) = P(A_2 \cap A_3) = 0.07 \]
\[ P(A_1 \cap A_3) = 0.01 \]

As mentioned above, in this paper I define systemic risk of degree \( r \) the probability of joint default of at least \( r \) financial intermediaries, \( P_r \). With only three banks, we obtain the following three measures of systemic risk:
\[ P_1 = P(A_1 \cup A_2 \cup A_3) \]
\[ P_2 = P((A_1 \cap A_2) \cup (A_2 \cap A_3) \cup (A_1 \cap A_3)) \]
\[ P_3 = P(A_1 \cap A_2 \cap A_3) \]

All these definitions involve unions and intersections of all three defaults event, and therefore they are all probabilities of order three, which is higher than the order of the information set presented above.

At first sight, one might think that if we observed all bond prices and all CDS spreads, thus learning \( P(A_i) \) and \( P(A_i \cap A_j) \) for each \( i \) and \( j \), we would be able to completely pin down the systemic probabilities \( P_1, P_2 \) and \( P_3 \). However, this is not the case: in general, an information set of order \( M \) cannot determine probabilities of order greater than \( M \). A simple graphical example of this is reported in Figure 2, which uses Venn diagrams to represent probabilities. In that Figure, the area of each event (circle) is the same across the two panels, so the marginal probabilities of defaults are the same. The same is true for the pairwise default probabilities. However, it is easy to see that \( P_3 \), the intersection of all three events, is positive in the top panel and zero in the bottom panel.

Knowledge of the low-order probabilities, however, allows us to put bounds on higher-order probabilities, and therefore on systemic default risk. While finding the upper bound for \( P_3 \) is immediate\(^{11}\) \((P_3 \leq 0.01)\), finding the other bounds is more complicated - and especially so when there are more than three banks in the financial sector. The exact way to obtain tightest bounds is the object of the rest of this Section. When applied to this example, it yields the following bounds:

\[ 0.45 \leq P_1 \leq 0.46 \]
\[ 0.13 \leq P_2 \leq 0.15 \]
\[ 0 \leq P_3 \leq 0.01 \]

\(^{11}\)Because we know \( P(A_1 \cap A_2 \cap A_3) \leq P(A_1 \cap A_3) = 0.01 \), and it is easy to see using Venn diagrams how this bound can be attained.
This simple example already shows one of the main points of this analysis: the set of bonds and CDS, when counterparty risk is taken into account, represent a rich information set that can be used to learn about systemic risk by making *no assumption* on the way these low-order probabilities aggregate at the higher-order level, but rather constructing bounds.

The example can be used to illustrate two additional concepts related to the measurement of systemic risk. The first one is that simply averaging the CDS spreads of financial institutions can lead to an erroneous measure of systemic risk - interpreted as probability of more than one bank defaulting together. Using the notation introduced above, such an index is:

\[
\frac{1}{6} \sum_i \sum_{j \neq i} z_{ij} = \frac{1}{6} \sum_i \sum_{j \neq i} (1 - R) \cdot (P(A_i) - (1 - S)P(A_i \cap A_j))
\]

\[
= \frac{(1 - R)}{3} \left[ \sum_i P(A_i) - (1 - S)P(A_1 \cap A_2) - (1 - S)P(A_2 \cap A_3) - (1 - S)P(A_1 \cap A_3) \right]
\]

If systemic risk increases (the joint default probabilities increase) while the marginal default probabilities remain the same (or do not increase enough), this index in fact decreases. The reason is that, because of counterparty risk, insurance gets worse but cheaper as a consequence of the increase in systemic risk, and therefore the average cost of insurance decreases. At least in some cases, then, this simple index fails to capture systemic risk properly.

The second idea that emerges from this example is the importance of using *all* the different prices available to construct the bounds, rather than using only the information contained in *average* bond and CDS spreads - which have been used before to construct alternative measures of systemic risk. While information on average default probabilities is enough to impose some bounds for the probability of systemic events, the additional restrictions given by the different prices can significantly improve on them. In the section below I prove that among all networks with the same average marginal and pairwise default probabilities, the widest bounds on systemic events (the least informative ones) are obtained when the network is symmetric, i.e. when all institutions have the same default probabilities. These bounds are in fact the same that are obtained if only average probabilities are observed. This tells us that the more *asymmetric* the financial network in terms of observed marginal and pairwise default probabilities, the highest the gain of using all the individual prices and spreads in terms of precision of the bounds.

Following on the example above, suppose that instead of observing all the marginal and pairwise default probabilities we just observed the average probabilities (this is a *partially aggregated* information set):

\[
\frac{P(A_1) + P(A_2) + P(A_3)}{3} = 0.2
\]

\[
\frac{P(A_1 \cap A_2) + P(A_2 \cap A_3) + P(A_1 \cap A_3)}{3} = 0.05
\]

In this case, the upper bound on systemic risk is \(P(A_1 \cap A_2 \cap A_3) \leq 0.05\). In fact, this bound is attained by a symmetric network configuration in which \(P(A_1 \cap A_2 \cap A_3) = P(A_i \cap A_j) = 0.05\) for all \(i\) and \(j\):
all joint default events (of two or three institutions) perfectly overlap at the upper bound. Using the full information set (that includes all bond and CDS prices) rather than the partially aggregated one allows us to reduce the upper bound on $P_3$ from 0.05 to 0.01. The same is true for the other definitions of systemic risk. The optimal bounds in this case are:

$$0.45 \leq P_1 \leq 0.50$$

$$0.05 \leq P_2 \leq 0.15$$

$$0 \leq P_3 \leq 0.05$$

which are wider than the full-information bounds.

3.2 General theory of the probability bounds

In this section, I show how to represent and solve the problem of constructing tightest bounds for probabilities of high-order events given a low-order information set. For now, I assume that we have already extracted the probability information set from the observed prices of traded securities. In Section 4 I discuss how to extract the marginal and pairwise default probabilities from the prices of bonds and CDS, and how limitations in the data I observe affect the estimation of the bounds.

Consider a finite set of basic events $\mathcal{A} = \{A_1, ..., A_N\}$, which in this paper I interpret as the default events of a set of $N$ financial intermediaries. The relation between low-order probabilities (probabilities of unions and intersections of a few events in $\mathcal{A}$) and higher-order ones (that involve many events in $\mathcal{A}$) has long been explored in mathematics. In particular, it is known that the knowledge of low-order probabilities is not enough to pin down completely high-order ones, but is enough to put bounds to them. Two famous results in this direction are Boole’s and Bonferroni’s inequalities, which state that:

$$P\left( \bigcup_i A_i \right) \leq \sum_i P(A_i)$$

and

$$P\left( \bigcap_i A_i \right) \geq \sum_i P(A_i) - (N - 1)$$

and therefore bound the probabilities that respectively at least one and all of the basic events occur, using only information about the individual probabilities of the basic events. These bounds are not tight, in the sense that tighter inequalities can be written based on the same information set (the set of all marginal probabilities of events in $\mathcal{A}$).

As discussed above, I define systemic event of degree $r$ the default of at least $r$ out of the $N$ intermediaries. Systemic events defined in this way are all of order $N$, because they involve unions and intersections of all the events in $\mathcal{A}$. I employ an estimation method that allows to obtain the tightest possible bounds
for the probabilities of systemic events given the available information set, which includes bond prices and CDS spreads. These reflect default probabilities of order 1 and 2, because they depend on the occurrence of one or at most two elements in \( A \). The approach is based on linear programming (LP), and consists in writing the bounds as the solution to a LP problem\(^\text{12}\), which is easy to solve numerically even for a very large scale, though analytical solutions are difficult to find.

To see how the LP approach works, for a sample space \( \Omega \) consider the finest partition of \( \Omega \) created by unions and intersections of the basic events \( A_1, \ldots, A_N \); call it \( V \). Then, the probability of each union or intersection of the basic events can be expressed as the sum of the probabilities of some events in \( V \). Since \( V \) contains exactly \( 2^N \) elements, it is possible to represent the probability space by a vector with \( 2^N \) elements, each corresponding to the probability of an elementary event in \( V \). Formally, the following proposition holds (see Boros and Prekopa (1989)):

**Proposition 1.** Call \( \mathcal{F} \) the \( \sigma \)-algebra generated by the finite set of events \( A_1, \ldots, A_N \) on a sample space \( \Omega \). Call \( V \) the finest partition of \( \Omega \) that is included in \( \mathcal{F} \). Then, \( V \) has \( 2^N \) elements, and any probability system on \( (\Omega, \mathcal{F}) \) can be represented by a vector \( p \in \mathbb{R}^{2^N} \), in the sense that \( \forall A \in \mathcal{F}, \exists I_A \subseteq \{1, 2, 3, \ldots, 2^N\} \) s.t. \( P(A) = \sum_{i \in I_A} p_i \).

In general, there are different vectors \( p \) that represent the same probability system. In this paper, I use the one constructed according to the following Proposition.

**Proposition 2.** For a set \( B \), call \( \overline{B} = \Omega \setminus B \), the complement of \( B \). For every integer \( i \) between 0 and \( 2^N-1 \), consider its binary representation \( b_i \), which consists of a vector of \( N \) numbers, each either 0 or 1. Construct \( p_{i+1} \) as follows:

\[
p_{i+1} = P(A_1^* \cap A_2^* \cap \ldots \cap A_N^*)
\]

where \( A_j^* = A_j \) if element \( j \) of \( b_i \) is 1, and \( A_j^* = \overline{A}_j \) if element \( j \) of \( b_i \) is 0. Then, \( p \) represents a probability system on \( \mathcal{F} \) in the sense of Proposition 1.

A simple example can help illustrate this Proposition. In the case of three banks, we have three basic events, \( A_1, A_2 \) and \( A_3 \) - the default events. The finest partition of the sample space obtained from unions and intersection of these events will have \( 2^3 = 8 \) elements. Figure 3 shows the 8 elements of this partition. From the Figure, it is evident how one can express the probability of any union or intersection of the \( A_i \)’s as the sum of the probabilities of a subset of these 8 elements. These 8 probabilities can be collected in a vector \( p \) with 8 elements, and therefore the probability of any event \( A' \) in \( \mathcal{F} \) can be represented as a product \( a'p \) for a certain vector \( a \). The consistency of the probability system \( p \) is assured by imposing \( p \geq 0 \) and \( i'p = 1 \), where \( i \) is a vector of ones.

The ordering of the elements of \( p \) is arbitrary, and Proposition 2 shows a way to construct a vector \( p \) that leads to a unique choice for the order of its elements. The \( i \)-th element of the vector \( p \) is obtained as follows. First, obtain the binary representation of the number \( i - 1, b_i \). For example,

\(^{12}\)See Kwerel (1975). A LP problem is a constrained maximization problem in which both the objective and the constraints are linear in the maximization variable.
Each of these vector can be interpreted as a vector of indicators of one of the three basic events. For example, $[011]$ represent the event in which $A_1$ does not occur, $A_2$ and $A_3$ occur. The element $i$ of $p_i$ will then be the probability of the event represented in this way by $b_i$. Therefore, we have:

$$
\begin{align*}
  p_1 &= Pr\{\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}\} \\
  p_2 &= Pr\{\overline{A_1} \cap \overline{A_2} \cap A_3\} \\
  p_3 &= Pr\{\overline{A_1} \cap A_2 \cap \overline{A_3}\} \\
  p_4 &= Pr\{\overline{A_1} \cap A_2 \cap A_3\} \\
  \vdots \\
  p_8 &= Pr\{A_1 \cap A_2 \cap A_3\}
\end{align*}
$$

This is precisely the ordering represented in Figure 3.

Propositions 1 and 2 imply that bounds on the probability of a (systemic) event $A'$ in $\mathcal{F}$ subject to constraints on low-order probabilities can be rewritten as a linear programming problem. In particular, the following Corollary holds:

**Corollary.** The upper bound for the probability of $P(A')$, i.e. the solution to:

$$
\begin{align*}
  \max P(A') \\
  \text{s.t.} \\
  P(A_i) &= a_i \\
  \vdots \\
  P(A_i \cap A_j) &= a_{ij}
\end{align*}
$$

can be found as the solution to the problem:

$$
\max_p c'p
$$
s.t.

\[
\begin{align*}
p & \geq 0 \\
i'p & = 1 \\
Ap & = b
\end{align*}
\]

for \(c, A, b\) depending only on the available information. The lower bound is obtained by solving the corresponding minimization problem.

**Proof.** The corollary is an immediate consequence of the fact that the probability of every union or intersection of events in \(A\) can be expressed as a product \(a'p\) for some \(a\).

As mentioned before, the definition of systemic event I employ is indexed by \(r\): “at least \(r\) institutions default” within a specified time period. This allows to capture systemic events of different degrees of severity, from the probability that at least one institution defaults (the union of all \(A_i\), \(r = 1\)) to the probability of default of all institutions (the intersection of all \(A_i\), \(r = N\)). Since these different events are all within the \(\sigma\)-algebra generated by the basic events \(A_1, ..., A_N\), their probability can be represented by the product of \(p\) with some vector \(c_r\): \(c_r'p\).

For the simple case of three banks reported above, it is easy to verify looking at Figure 3 that the probabilities \(P_r\) of at least \(r\) institutions defaulting can be expressed as

\[
\begin{align*}
P_1 & = [01111111] \cdot p \\
P_2 & = [00010111] \cdot p \\
P_3 & = [00000001] \cdot p
\end{align*}
\]

and that the constraints can be rewritten as:

\[
\begin{align*}
P(A_1) & = [00001111] \cdot p = a_1 \\
P(A_1 \cap A_2) & = [00000011] \cdot p = a_{12}
\end{align*}
\]

and so on. These constraints can then be collected in a matrix \(A\) and a vector \(b\), obtaining the representation above.

The problem derived in the Corollary is a standard LP program and while difficult to solve analytically, it is easy to solve numerically even as the scale of the problem gets large. Besides, the linearity of the problem guarantees that the global optimum is always found when solving it numerically. Finally, Farkas’ lemma guarantees that it is always possible (and numerically feasible) to prove whether the system has no solution\(^{13}\), which is important to confirm the convergence of the numerical algorithm.

\(^{13}\)Farkas’ lemma states that one and only one of the following is true: 1) There exists an \(x \in R^N\) such that \(Ax = b\) and \(x \geq 0\). 2) There exists a \(y \in R^M\) such that \(A'y \geq 0\) and \(b'y < 0\). Finding a solution to the second system of inequalities immediately proves that the probability bounds problem is infeasible given the constraints.
3.3 Properties of the bounds

Because the probability bounds are the solution to a LP problem, it is difficult to fully characterize them algebraically. However, some important properties can be derived from the linear structure of the problem. First, I prove in Proposition 3 that if the maximization problem is symmetric to the ordering of the default events (which is for example the case if we only observe average CDS and bond information), then the network configuration that attains the bounds will be symmetric - all banks have the same probability of default and all pairs of banks have the same pairwise probability of default. Second, I show that as a consequence of this result, I can apply some results from the LP literature (e.g. Boros and Prekopa (1989)) to derive closed form solutions for the bounds when the network is symmetric. In particular, I can study how the width of the bounds varies as the underlying low-order probabilities of default change.

3.3.1 Symmetry of the probability system

**Definition.** Consider the vector \( p \in \mathbb{R}^{2^N} \) representing a probability system on the \( \sigma \)-algebra generated by the basic events \( A_1, ..., A_N \), as in Propositions 1 and 2. Consider a permutation \( J \) of the indices of the basic events: \( A_{J_1}, ..., A_{J_N} \), and call \( J \) the set of permutations. Define \( p_J \in \mathbb{R}^{2^N} \) the vector representing the probability system generated by \( A_{J_1}, ..., A_{J_N} \) that corresponds to \( p \), constructed as in Proposition 2.

**Definition.** A linear combination of the elements of \( p \) defined by the vector \( c \) is *symmetric* with respect to the generating events \( A_1, ..., A_N \) if \( c' p = c' p_J \forall J \in J \).

For example, take two events \( A_1 \) and \( A_2 \). A vector \( p \) representing the probability system constructed as in Proposition 2 would have 4 elements, corresponding to:

\[
\begin{align*}
P(\overline{A}_1 \cap \overline{A}_2) &= p_1 \\
P(\overline{A}_1 \cap A_2) &= p_2 \\
P(A_1 \cap \overline{A}_2) &= p_3 \\
P(A_1 \cap A_2) &= p_4
\end{align*}
\]

In this case, only one additional permutation of the generating events is possible, \( J = \{2, 1\} \), and we have:

\[
\begin{align*}
p_{J1} &= p_1 \\
p_{J2} &= p_3 \\
p_{J3} &= p_2 \\
p_{J4} &= p_4
\end{align*}
\]
An example of symmetric weighting vector $c$ is the one corresponding to the probability of the union of the events, $c = [1 \ 1 \ 0 \ 1]'$, since $c'p = c'p_J = P(A_1 \cup A_2)$.

**Definition.** A probability system $p$ is symmetric if every event in $V$ (the finest partition of the sample space generated by the basic event) has the same probability in all permutations of the generating events.

For example, with three generating events ($N = 3$), the probability system is symmetric if $P(A_1) = P(A_2) = P(A_3)$ and $P(A_1 \cap A_2) = P(A_1 \cap A_3) = P(A_1 \cap A_3)$.

**Definition.** A linear programming problem

$$\text{max } c'p$$

$$\text{s.t. } Ap \leq b$$

is symmetric if $c$ and all rows of $A$ are symmetric with respect to the generating events $A_1, ..., A_N$.

We can now state the following proposition:

**Proposition 3.** Suppose that the probability bounds correspond to a symmetric LP problem. Then, the bounds are attained by a symmetric probability system.

**Proof.** See Appendix. \qed

**Corollary.** The bounds on systemic events of the type “at least $r$ institutions default” given a symmetric constraint set (for example, constraints on the average marginal and pairwise default probabilities) are attained by a symmetric probability system.

The bounds obtained in a symmetric network in which we observe all marginal and pairwise probabilities will always be at least as wide as those obtained in an asymmetric network with the same averages of the low-order probabilities. The difference between the bounds obtained in the two cases captures precisely the extent to which asymmetry in the network shape affects the probability of systemic events.

### 3.3.2 Width of the bounds

The Corollary to Proposition 3 also implies that the analytical formulation of the bounds derived by Boros and Prekopa (1989) for the case in which only the average marginal and pairwise default probabilities are known applies to the case in which all low-order probabilities are observed, as long as the network is symmetric with respect to the latter. This is an interesting benchmark to understand some of the properties of the bounds. The following Proposition holds:

**Proposition 4.** Denote with $p_r$ the probability of the occurrence of at least $r$ events. If the probability system is symmetric with respect to marginal and pairwise default probabilities, i.e.

$$P(A_i) = q_1 \ \forall i$$

19
\[ P(A_i \cap A_j) = q_2 \forall i, j \]

then the upper and lower bounds for \( p_r \), \( r \geq 3 \), out of \( N \) events, have the following properties:

- For given \( q_1 \)
  - if \( q_1 < \frac{r-1}{N} \), the lower bound is 0 for low \( q_2 \) and is increasing in \( q_2 \) for higher \( q_2 \).
  - if \( q_1 > \frac{r-1}{N} \), the lower bound is first decreasing and then increasing in \( q_2 \).

- For given \( q_1 \)
  - if \( q_1 < \frac{N-2}{N(N-2) - r+1} r \), the upper bound is first increasing and then decreasing in \( q_2 \).
  - if \( q_1 > \frac{N-2}{N(N-2) - r+1} r \), the upper bound is first 1 and then decreasing in \( q_2 \).

- Given \( q_1 \), the width of the bounds is 0 for the lowest possible \( q_2 \), then increases with \( q_2 \) and then decreases to 0 for \( q_2 = q_1 \). Therefore, there is a point of maximum width in the interior of the space for \( q_2 \).

- Similar results hold when \( q_1 \) varies and \( q_2 \) is held fixed.

**Proof.** See Appendix. \( \square \)

Proposition 4 implies that the tightness of the bounds on systemic risk varies in a very precise way with changes in the low-order risks. When pairwise default probabilities \( q_2 \) are either very high or very low compared to the marginal default probabilities \( q_1 \), the structure of the network is pinned down very precisely and there is little uncertainty about systemic default risk, based on bond and CDS prices alone; different parametric models that aggregate in different ways this low-order information will agree on an estimate of systemic risk. For intermediate values of \( q_1 \) and \( q_2 \), however, low-order probabilities are less informative about systemic events. The scope for modeling assumptions in aggregating low-order probabilities is greater.

Therefore, if direct information on systemic risks is scarce and agents learn about them by combining low-order information with some parametric model, the width of the bounds captures the scope for disagreement based on modeling assumptions. Starting from a situation of low risks (low \( q_1 \) and \( q_2 \)), increases in the marginal and pairwise risk correlations will coincide with an increase in the scope for disagreement, at least up to the point where the bounds narrow again (and everybody agrees on the severity of systemic risks). This might have important consequences for the stability of the financial system, because it allows for more extreme disagreement about systemic risks even for fully rational agents facing the common information set, precisely when low-order risks increase.
4 Probability bounds: implementation

The method presented in the previous Section allows to estimate the bounds on the probability of systemic events if one could directly obtain from bond prices and CDS spreads all marginal and pairwise default probabilities for all banks and all pairs of banks. In practice, however, when working with actual pricing data, estimating the bounds requires additional steps and is subject to some limitations.

First, in order to extract marginal and pairwise default probabilities from observed prices, I need to specify a pricing model for bonds and CDSs, that takes into account not only default risk, but also other important determinants of prices. Several studies (such as Huang and Huang (2003) or Longstaff et al. (2005)) have established that liquidity premia in the bond market are, together with default risk, a major component of the yield spread. I will therefore pay particular attention to the liquidity factor. In addition, the implementation of the bounds is affected by the availability of CDS data. The theoretical bounds presented above condition on all pairwise default probabilities, and could be estimated directly if I observed, for each pair of dealers $i$ and $j$, the price at which $j$ sold CDS protection against $i$’s default. Unfortunately, I only observe an average of the quotes reported by different dealers, and therefore the bounds I can estimate condition on a smaller information set, which captures average counterparty risk. I now discuss each of these issues separately.

4.1 Pricing bonds

To price bonds, I use a simple pricing model of the reduced-form class, as in Duffie (1998), Lando (1998), Duffie and Singleton (1997, 1999), and Hull and White (2000, 2001), with constant risk-neutral hazard rates of default.

Three elements are crucial in determining the price of a bond: credit risk, the recovery process and the liquidity premium. Assume that the recovery process is independent of all other processes, and call the expected recovery rate $R$. Call $T$ the maturity of the bond and $r^F_t$ the riskless rate process. The reduced-form approach consists in specifying a risk-neutral default hazard process $h_t$, the risk-neutral probability of default between $t$ and $t + 1$ conditional on survival until $t$. In addition, following Duffie (1999) I incorporate a potentially time-varying liquidity process $\gamma_t > 0$. This process is modeled as a per-period cost of holding the bond. In a later section I discuss how this general way of modeling liquidity connects to more practical measures of liquidity, such as margin requirements.

Call the $r^F_{t,T}$ the realized return of the short-term risk-free security between $t$ and $T$, s.t.

$$(1 + r^F_{t,T}) = \prod_{s=t+1}^{T} (1 + r^F_s)$$
and $G(t, T)$ the probability of survival up to $T$ under a certain realization of hazard rates of default, i.e.:

$$G(t, T) = \prod_{s=t+1}^{T} (1 - h_s)$$

The time $t$ price of a risk-free zero-coupon bond of face value $1$ at time $T$ is:

$$\delta(t, T) = E^Q_t \left[ \frac{1}{1 + r_{t,T}^F} \right]$$

where $E^Q$ indicates the expectation taken under the risk neutral probability measure.

The price of a liquid bond $j$ issued by firm $i$ of face value $1$, maturity $T_{ij}$, coupon rate $c_{ij}^j$ and recovery equal to a fraction $R_{ij}^j$ of the value of a Treasury zero-coupon bond of comparable maturity\footnote{All results are robust to the assumption that all bonds of equal seniority recover the same fraction of the face value of the bond.} is:

$$B_{ij}^j(t, T_{ij}^j) = E^Q_t \left[ \sum_{s=t+1}^{T_{ij}} \frac{G^i(t, s)}{1 + r_{t,s}^F} c_{ij}^j + \frac{G^i(t, T_{ij}^j)}{1 + r_{t,T}^F} + \sum_{s=t+1}^{T_{ij}} \frac{G^i(t, s - 1) h_s^i}{1 + r_{t,T}^F} R_{ij}^j \right]$$
$$+\delta(t, T^{ij})(1 - h_t^{ij})^{T^{ij}}(1 - \gamma_t^{ij})^{T^{ij}} + R \left( \sum_{s=t+1}^{T^{ij}} \delta(t, T)(1 - h_t^{ij})^{s-1}(1 - \gamma_t^{ij})^{s-1} h_t^{ij} \right)$$  \hspace{1cm} (1)$$

Before tackling the problem of calibrating the process $\gamma_t^{ij}$ (later in this Section), it can be useful to discuss a possible interpretation for this variable. The role of $\gamma_t^{ij}$ in the bond pricing formula is to capture in a reduced-form way all the different elements that result in a liquidity discount in bonds, which can arise for a variety of reasons (such as funding costs, search costs and other transaction costs, or asymmetric information). Funding costs, in particular, are an appealing motivation for modeling the liquidity process because they are known to have been especially relevant during the recent crisis, and because it is easier to calibrate them, since they are directly related to observable variables like the collateral requirements and the cost of capital.

Garleanu and Pedersen (2010) present a model in which liquidity discounts arise because some market participants are required to use part of their own equity in purchasing the assets. In particular, they are required to post a margin $m_t^{ij}$ on security $j$ of firm $i$ at time $t$. Because this uses up part of their own capital, they require an additional return that is proportional to the product of $m_t^{ij}$ and $\psi_t$, the shadow cost of funds at time $t$. This, in the presence of a group of traders for which the financing constraint is binding, leads to an adjusted CCAPM of the form:

$$E_t[R_{t+1}^{ij} - R_{t+1}^{f}] = -\frac{\text{Cov}_t(M_{t+1}, R_{t+1}^{ij} - R_{t+1}^{f})}{E_t[M_{t+1}]} + m_t^{ij} x_t \psi_t$$

where $M_{t+1}$ is the CCAPM stochastic discount factor and $R_{t+1}^{ij}$ is the return on the bond, which includes the potential liquidity discount that might arise in the future. The liquidity discount also depends on $x_t$, the proportion of liquidity-constrained agents in the economy. Note that since all bonds I consider have the same seniority, we can assume that $m_t^{ij} = m_t^{i}$ for all the bonds issued by the same firm.

If we keep the independence assumptions among the different processes introduced earlier, and again we assume that at each time $t$ bonds are priced as if the hazard rate $h_t$ as well as the liquidity-related variables $m_t^{ij}, x_t$ and $\psi_t$ will be constant in all future periods $t + s$, we can rewrite this expression as:

$$B(s, T) = (1 - h_t)E_t^Q[c + B(s + 1, T)|\text{no default}](1 - (m_t^{ij} x_t \psi_t)) E_s[M_{s+1}]$$

assuming for simplicity a recovery rate of 0. It is easy to see that the liquidity premium affects the bond price in a proportional way, similarly to the liquidity process $\gamma_t^{ij}$. To be precise, under these assumptions $\gamma_t^{ij}$ is approximately equal to $E[M_{s+1}] m_t^{ij} x_t \psi_t^{15}$.

In light of this, at a first approximation, the liquidity component $\gamma_t^{ij}$ can be interpreted as a time-varying shadow cost of the capital that needs to be put as margin on the bond issued by firm $i$. Variation of $\gamma_t^{ij}$ over time can be attributed to changes in the margin requirements specific to firm $i$ or to variations in the economy-wide weighted shadow cost of capital, $x_t \psi_t$.

\hspace{1cm} \text{15Apart from a term } E_s[M_{s+1}] - E[M_{s+1}] \text{ which is small and not very volatile.}


4.2 Pricing CDSs

As explained in Section 2, a potentially important component of the spread of a CDS is counterparty risk. Counterparty risk arises because in some states of the world the protection seller cannot pay the buyer the full amount owed. A fraction of that amount can still be recovered thanks to collateralization and to the seniority of CDS claims in bankruptcy relative to junior claims.

As with bonds, I discretize the model to one month intervals, and I assume that both marginal and joint hazard rates are constant from the perspective of the time of the pricing until the maturity of the CDS (5 years). I assume that the payoff of the CDS for the following month is as follows. If the seller does not default within the month but the reference entity defaults, the payment is made in full. Therefore, a month is considered an amount of time sufficient for the seller to establish whether to default or not on the CDS obligation, given that the reference entity defaulted. If the seller defaults within the month but the reference entity does not, the contract theoretically terminates with either a positive or a negative value (depending on whether the default probability of the reference entity increased or decreased relative to when the contract was written). Here I assume that the expected value of the contract conditional on the reference entity surviving until the next month is zero. This is consistent with the assumption that if the default of one institution increases the default hazard of the other one, the effect plays out immediately but tends to decrease over time; it is also consistent with the bond pricing model discussed above, in which the probability of default of each institution is constant conditional on survival. In any case, the pricing formula remains approximately the same as long as the hazard rate of default of the reference entity, conditional on surviving until the next month, remains of the same order of magnitude as it was before the default of the counterparty. In this case, the effect on the price of the CDS is of the order of magnitude of the square of the CDS spread, which is very small.

Finally, if both the seller and the reference entity default in the same month, I assume that the two defaults happen in a connected way and only an amount $S$ of the full payment is recovered - this case corresponds to the double default case, in which counterparty losses are important. Again, the pricing formula remains approximately the same if conditional of both firms defaulting in the same month, the default of the seller induces an immediate jump in the order of magnitude of the probability of default of the other institution for a certain amount of time, before it defaults: double default does not need exact simultaneity of the defaults. Note that different assumptions about the average change in probability conditional on both banks defaulting in the same month can be mapped into different recovery rates in case of double default, $S$. Robustness to assumptions about $S$ is explored in Section 7.

Calling $P(A_i)$ the (constant) monthly default probability of institution $i$, $P(A_i \cap A_j)$ the probability of joint default during each month, and under the assumptions that these hazard rates are constant over time and independent of the risk-free rate process, the discretized CDS pricing equation can be written as:
\[
\sum_{s=t}^{T-1} \delta(t, s-1)(1 - P(A_i \cup A_j))^{s-1} z_{ji} = \\
\left[ \sum_{s=t+1}^{T} (1 - P(A_i \cup A_j))^{s-1}(P(A_i) - (1 - S)P(A_i \cap A_j))\delta(t, T)(1 - R) \right]
\]

where \( P(A_i \cup A_j) = P(A_i) + P(A_j) - P(A_i \cap A_j) \), and \( z_{ji} \) is the spread of the CDS written by \( j \) to insure against \( i \)'s default.

The left-hand side of the formula represents the present value of payments to the protection seller: they only occur as long as neither a credit event occurred nor the counterparty defaulted. The right-hand side represents the potential payment in case of default. In each period, conditional on both firms surviving until then, there is a probability \( P(A_i) - P(A_i \cap A_j) \) that the reference entity defaults while the counterparty has not defaulted, so that the payment of \( \delta(s, T)(1 - R) \) is made in full. With probability \( P(A_i \cap A_j) \), there is a double-default event, and thus only a fraction \( S \) of that payment is recovered. Note that if only the counteparty defaults the contract ends with zero value due to the assumption of constant hazard rates\(^{16}\).

After a linear approximation derived and discussed in the Appendix, it is possible to rewrite the spread as:

\[
z_{ji} = (P(A_i) - (1 - S)P(A_i \cap A_j)) \frac{T\delta(0, T)(1 - R)}{\sum_{s=1}^{T} \delta(0, s - 1)} \tag{2}
\]

Note that this equation is a restriction on the marginal and pairwise default probabilities of \( i \) and \( j \), and as such it could be used to obtain a constraint on \( P(A_i \cap A_j) \) for given \( P(A_i) \). However, it can also be imposed directly as a constraint in the LP program presented above, since it is linear in the marginal and pairwise default probabilities.

While the model I use in the paper uses a simple discrete-time framework, it would be possible in theory to build continuous-time models of CDS prices that take into account the exact dynamics of events and the timing of defaults. Unfortunately, the availability of CDS pricing data is so limited that it would not allow extracting joint default processes in a model with several free parameters. Note that several models of joint default that appear in the literature are based on the assumption that default intensities are independent conditional on the realization of a state vector; in these models, counterparty risk at short horizons is small by construction, because it is proportional to the product of the marginal default probabilities. Therefore, these models would not be the most appropriate ones for the case in which defaults are not independent even at short horizons. A valid alternative would be a model of correlated default intensities, as introduced by Jarrow and Yu (2001): there, defaults are independent over very short horizons of time, but the default of one institution increases the default intensity of the others. While I do

\(^{16}\)See discussion in Section 2. Section 7 presents a case in which the hazard rate is time-varying, and in that case I assume the payment is made in full.
not explicitly use this model in my empirical analysis, my discrete-time pricing formulation is compatible with a model in which defaults within a certain period of time (a month) are correlated due to spillovers from one institution to the other. However, conditional on one institution surviving for enough time after the default of the other, the spillover effect on the default intensity of other banks is small.

4.3 Implementation of the bounds: liquidity assumptions

Once the pricing models for bonds and CDSs are specified, the implementation of the bounds requires two further steps. The first one is dealing with the presence of the bond liquidity process $\gamma_{it}$, which is unobservable but a crucial determinant of bond prices. The second, discussed in the next section, is modifying the linear programming problem to take into account the limitations of the CDS data.

The general bounds described above would theoretically require to obtain an estimate of $P(A_i)$ for each $i$ from bond prices. Obviously, this can be obtained if one is willing to estimate $\gamma_{it}$ for each firm. While directly estimating it is extremely difficult, it is at least possible to obtain plausible lower bounds for it, which translate into upper bounds for the marginal probabilities $P(A_i)$. One can then modify the LP problem by using these upper bounds as constraints.

We can start by decomposing the (unobservable) liquidity process $\gamma_{it}$ into a fixed firm-specific component $\alpha_i$, and a time-varying component $\lambda_t$ common to all senior unsecured bonds issued by LFIs:

$$\gamma_{it} = \alpha_i \lambda_t$$

where $\lambda_t$ is a latent variable normalized to be 1 on average during 2004 (the beginning of the sample), so that $\alpha_i$ captures the average liquidity component of each bank in the early period. Note that this formulation is flexible enough to capture constant differences among firms in my sample, as well as changes in margins and other liquidity-related costs that are common to the firms in the sample even though they are different from the rest of the economy.

This decomposition, together with the interpretation of $\gamma_{it}$ discussed above, suggest three possible ways to impose a plausible lower bound on $\gamma_{it}$ for the financial institutions in my sample. The first approach just requires that the liquidity premium for bonds should not be negative:

$$\gamma_{it} \geq 0$$

The idea behind the second approach is that we can use the early part of the sample (2004) to identify $\alpha_i$, because in this period counterparty risk was considered to be essentially zero. Then, using bond prices together with CDS prices allows us to perfectly identify $\gamma_{it}$ during this period. The average over this period will identify $\alpha_i$, while $\lambda_t$ is on average 1 (and has very low variance). Given that we can plausibly assume that liquidity premia did not decrease during the crisis, neither for the economy at large nor in particular
for financial firms, we can impose the lower bound:

\[ \gamma_i^t \geq \alpha_i \]

A third, more sophisticated approach tries to obtain a time-varying lower bound for the liquidity process \( \gamma_i^t \), by comparing the bonds issued by the financial institutions in the sample to other non-financial institutions with high credit ratings and therefore likely similar margins and cost of funding. A CDS written by a financial institution on a safe non-financial firm is much less likely to contain counterparty risk, meant as the *risk of double default*. Under this assumption, I proceed as follows. For a set \( J \) of nonfinancial firms with high credit rating, I estimate \( \gamma^j_t \) using bond yield spreads and CDS spreads together and assuming no counterparty risk\(^{17}\). I then decompose it as

\[ \gamma^j_t = \alpha_j \lambda_t^* \]

therefore allowing the component common to nonfinancial firms (\( \lambda_t^* \)) to be different from the common component of financial firms, \( \lambda_t \). Assuming that \( \gamma^j_t \) is observed with noise \( \epsilon_t^j \) uncorrelated with \( \alpha_j \) and \( \lambda_t^* \), we observe:

\[ \hat{\gamma}_t^j = \gamma_t^j + \epsilon_t^j \]

we can then estimate using OLS the series \( \lambda_t^* \) for each \( t \) (again, normalizing s.t. \( \lambda_t^* \) is 1 on average in 2004). Since this series captures the cost of funds as well as the margin requirement of the bonds of these non-financial institutions (relative to the pre-crisis level), for a high credit rating group \( J \) of nonfinancial firms it is reasonable to assume that \( \lambda_t \geq \lambda_t^* \). In other words, the liquidity component common to financial firms was, during the crisis, at least as high as the component common to nonfinancial firms.

This allows to impose the time-varying lower bound on \( \gamma^i_t \):

\[ \gamma^i_t \geq \alpha^i \lambda_t^* \]

Once a lower bound \( \gamma^i_t \) is obtained in this way for each \( i \), from the bond pricing equation we immediately obtain an upper bound on \( P(A_i) \), which is the value of \( P(A_i) \) that is estimated from the bonds when \( \gamma^i_t = \gamma^i_t \) : call this function \( h_i(\gamma^i_t) \). This result follows immediately from the fact that \( h_i(\gamma^i_t) \) is decreasing in \( \gamma^i_t \).

We can then modify the maximization problem to find the bounds on systemic risk by replacing at each \( t \) the constraints \( P(A_i) = \alpha_i \) with the inequality constraints (one for each \( i \))

\[ P(A_i)_t \leq h_i(\gamma^i_t) \]

This allows to preserve the LP formulation in computing the bounds. On the other hand, it will result

\(^{17}\)Assuming independence yields very similar results, since the order of magnitude is the square of the CDS spread and therefore extremely small.
in wider bounds, since equalities have been replaced by inequalities.

An important caveat with the use of these liquidity assumptions is that in some periods, and for some banks, the upper bound on the marginal default probability \( h_i(\gamma_i^t) \) might be lower than the minimum probability consistent with CDS spreads (the one obtained under the assumption that counterparty risk is nonnegative). In other words, after removing the component of the yield spread due to liquidity, the effective bond/CDS basis can become positive (i.e. the CDS spread is higher than the yield spread). Naturally, the more stringent the assumptions on liquidity are, the more often this can happen. For example, if the liquidity process is set so that the average basis in 2004 is zero, this will happen in about half of the cases up until 2007 (the median is close to the mean). I interpret these as cases in which counterparty risk is essentially zero. For the purposes of the estimation, the institution is considered isolated from the rest of the network at that time (the joint default probability with any other bank is 0) and its marginal default probability is set at the one implied by CDS spreads in the absence of counterparty risk. All the results hold true if instead the default probability is set at \( h_i(\gamma_i^t) \).

### 4.4 Availability of CDS data

The second factor to take into account when implementing the LP problem is the fact that for each institution \( i \) I do not observe the spread written on its bond by every other institution \( j \), but only an average of the quotes provided by the \( N - 1 \) counterparties:

\[
\bar{z}_i = \frac{1}{N - 1} \sum_{j \neq i} z_{ji}
\]

This means that when I compute the bounds, instead of the set of constraints of the type (2), one for each pair \( i \neq j \), I can only impose the constraint

\[
\bar{z}_i = \left[ P(A_i) - (1 - S) \left( \frac{1}{N - 1} \sum_{i \neq j} P(A_i \cap A_j) \right) \right] \frac{T \delta(0, T)(1 - R)}{\sum_{s=1}^{T} \delta(0, s - 1)}
\]

for each \( i \). It is important to note that this constraint assumes that the spread \( \bar{z}_i \) is obtained by averaging across the spreads quoted by the other \( N - 1 \) institutions in the group considered. Unfortunately I do not observe exactly which dealers contributed to the quotes, nor do I observe the exact weighting scheme employed, which presumably gives more weight to more active institutions. My solution is to consider a group of firms that most likely represents the sample from which the quotes come from. Since this market is very concentrated, and the top 10 firms alone account for about 90% of the protection sold, I believe that including the top 14 dealers should make sure that the average spread reflects the average counterparty risk of this group of dealers. Besides, since all of these institutions are very active in the CDS market, equal weighting seems a reasonable approximation of the true weighting scheme. Of course, it is possible that the spread partly reflects quotes obtained from financial institutions outside the group.
I consider, or that some of the dealers in the group do not post quotes at all times. In both cases, I would likely underestimate counterparty risk. In the former case, because quotes may be obtained from smaller institutions for which the recovery rate of the CDS in case of double default could be lower. In the latter case, because if the institutions that are not posting a quote are the riskier ones, the average spread observed would be biased upwards. However, as long as these problems affect only a few institutions at a time, the effect on the average spread should be small.

4.5 Recovery rates

To construct the bounds on systemic risk, I need to specify expected recovery rates. The baseline expected recovery rate for senior unsecured bonds ($R$) is 30%. For CDSs, I assume a recovery rate of 30% in case of double default ($S$), and of 100% in case of default of the seller alone. This implies that, if double default occurs, CDS and senior unsecured bond claims on a firm are equivalent. Since CDSs, like the other derivative contracts, have the additional right to seize the collateral posted for them, and since at least a fraction of CDS contracts have a collateral agreement, this expected recovery rate might in practice be higher. For the reasons discussed in Section 2, however, it is unlikely that collateral can provide sufficient protection against the double default case, so even taking collateralization into account, the recovery rate in that case would likely be not much higher than that of senior unsecured bonds. Robustness to assumptions on $R$ and $S$ is explored in Section 7.

4.6 Feasible bounds

The analysis presented above allows us to reformulate the maximization problem to obtain bounds on systemic risk that take into account all these factors. The optimal bounds that can be computed given the assumptions discussed above are as follows:

\[ \max P_r \]

s.t.

\[ P(A_i) \leq h_i(\gamma_i) \forall i \]  \hspace{1cm} (3)

\[ \left[ P(A_i) - (1 - S) \left( \frac{1}{N - 1} \sum_{i \neq j} P(A_i \cap A_j) \right) \right] \frac{T \delta(0, T)(1 - R)}{\sum_{s=1}^{T} \delta(0, s - 1)} = z_i \forall i \]  \hspace{1cm} (4)

which can be represented in linear form as:

\[ \max p \epsilon' p \]

s.t.

\[ p \geq 0 \]

\[ i' p = 1 \]

\[ Cp \leq d \]  \hspace{1cm} (5)
\[ Ep = f \]  

where the constraint (5) corresponds to the set of constraints (3), and the constraint (6) corresponds to (4). These bounds can then be computed separately at each \( t \) using the cross-section of bond and CDS prices, as well as the series for \( \gamma_t^i \).

5 Data

5.1 Choice of the set of intermediaries

The focus of this paper are the systemic events that affect the core of the global financial network, represented by the group of largest and most connected financial institutions. Motivated by the literature on contagion in financial networks\(^{18}\), which emphasizes the role of the gross risk exposures of intermediaries for the propagation of shocks, I focus on the set of dealers with the highest gross exposures to the CDS market. In particular, I choose a group of 14 institutions that includes the largest American and European broker/dealers, but excludes large firms (like AIG) which have higher net exposure to credit risk but lower gross exposure. The list contains 8 American and 6 European investment banks, commercial banks and broker/dealers: Bank of America, Citigroup, Goldman Sachs, Lehman Brothers, JP Morgan, Merrill Lynch, Morgan Stanley, Wachovia, Abn Amro, Bnp Paribas, Barclays, Credit Suisse, Deutsche Bank, UBS.

Several sources confirm that this group of banks is really at the center of the financial network, at least as far as CDS exposures are concerned. Fitch Ratings, referring to different years (2006, 2008), reports that the top 5 dealers accounted for more than 80% of the value of outstanding CDS contracts. Credit Derivatives Research claims that the group of 14 institutions mentioned above (plus HSBC, for which bond data were too scarce to include in the sample) covers 90% of the volume of CDS protection sold. Not only are these firms the largest issuers and buyers of CDS protection: they are also consistently among the top entities on which protection is sold. In other words, these institutions are at the same time selling protection to each other against external default events, and exchanging protection against the default of other members of the same group. For example, Fitch reports that the top 5 sellers of CDS protection also appear in the set of top 20 reference entities.

The data cover, with daily frequency, the period from January 2004 to June 2010.

5.2 Data description

5.2.1 Corporate Bonds

As discussed in Section 4, an important step of the construction of the bounds is the estimation, from bond prices, of the risk-neutral probability of default given a lower bound for the liquidity process: $h_i(\gamma_i^t)$. This probability represents the upper bound on the probability $P(A_i)$.

For each of the 14 institutions considered, I search Bloomberg for senior unsecured zero and fixed coupon bonds with maturity less than 10 years\(^{19}\). I exclude callable, putable, sinkable, and structured bonds, since their prices reflect the value of the embedded options. I remove all bonds for which I have price information for less than 5 trading days. I consider bonds denominated in six main currencies: USD, Euro, GBP, Yen, HKD, CHF. I discount the cash flows using the corresponding Treasury curves (see Section 7 for a discussion about using bonds and CDSs denominated in different currencies to estimate the bounds). Since the availability of Bloomberg data on European bonds is fairly limited, I integrate it with bond pricing data from Markit, whenever it adds at least 5 observations.

For every trading day $t$, I estimate the risk-neutral probability $h_i^t(\gamma_i^t)$ separately for each firm, using the cross section of bonds issued by firm $i$ which are still outstanding at time $t$. I employ the simple pricing model described in Section 4, and estimate the hazard rate using least absolute deviations to reduce the impact of outliers. All the main results are robust to the use of OLS.

Table 1 reports some statistics on the availability of bond data. In the first column, for each institution, we can see the average daily number of bonds I am able to use for the estimation. For example, the default probability for Bank of America $h_i^t$ is estimated using on average 32 bonds each day. The next columns break down this number by year. For some European dealers, bond data is missing in a few periods. These dealers will still be part of the empirical analysis below, but they will be excluded for the periods with missing bond data.

5.2.2 Risk-free rate

As the reference risk-free rate, I use US and European government bond yields, constructed from the generic Bloomberg yield curves, as is customary for reduced-form models. An alternative would be to use swap rates, which are less affected by tax and liquidity issues. Their use for investment-grade bonds is advocated by Howeling and Vorst (2005). However, swap rates contains some counterparty risk, and, more importantly, they are indexed to LIBOR (see Sundaresan (1991) and Duffie and Huang (1996)). Because LIBOR is the rate on unsecured loans between banks, it cannot be considered risk-free, especially in the context of systemic risk in the financial system. Therefore, I prefer to employ government bonds\(^{19}\). Given that the maturity of CDSs is 5 years, it would make sense to use outstanding bonds of remaining maturity close to 5 years when comparing bonds and CDSs. While the results are robust to other maturity brackets around 5 years - for example (2,8) and (4,6), this reduces the number of institutions I can use for the estimation. Using (0,10) allows to keep all 14 institutions for most of the sample.

\(^{19}\)Given that the maturity of CDSs is 5 years, it would make sense to use outstanding bonds of remaining maturity close to 5 years when comparing bonds and CDSs. While the results are robust to other maturity brackets around 5 years - for example (2,8) and (4,6), this reduces the number of institutions I can use for the estimation. Using (0,10) allows to keep all 14 institutions for most of the sample.
in my analysis. All the results are actually strengthened if swap rates are used instead of Treasury rates. This should not be surprising since swap rates are generally higher than Treasuries, which means that the marginal default probability extracted from bond prices is lower, and therefore counterparty risk has to be lower as well. Besides, the difference between the two rates kept increasing slowly until Lehman’s collapse, and had a sharp increase only after that. The time series of this difference if anything strengthens the result of low systemic risk before the end of 2008.

5.2.3 Credit Default Swaps and the bond/CDS basis

Since Credit Default Swaps are derivatives traded over the counter, they are in general not standardized. However, the 5-year CDS based on the ISDA format (which standardizes credit event definitions and collateral requirements) has emerged as the reference contract, which allowed it to achieve high liquidity. For this CDS, Markit reports mid-market quotes that are obtained by averaging the quotes reported by different dealers - euro-denominated for European banks and dollar-denominated for US banks. The series contain a few missing values, which are filled by interpolation. All the results are robust to the use of CDS data from Bloomberg (obtained from CMA)\(^\text{20}\).

Table 1 reports summary statistics on CDS spreads. While CDS spreads between 2004 and 2010 have usually been quite low, in the order of 50bp, they reached levels higher than 1000bp in some periods. Figure 4 reports the time-series of CDS spreads for some banks in the sample (thin line)\(^\text{21}\). From the figure, it is easy to see that while movements in CDS spreads seem to be highly correlated across dealers, there is a certain degree of heterogeneity, especially since 2007. The highest spreads are reached in the few months after Lehman’s collapse, and in the first half of 2009, but not for all banks at the same time.

As discussed in Section 1, in the absence of counterparty risk and liquidity premia, CDS spreads, the risk-free rate and bond yields are related by an approximate arbitrage relation\(^\text{22}\). The deviation from this arbitrage, and in particular the difference between the CDS spread and the corresponding corporate yield spread is the basis of the CDS trade, and has been extensively studied in the literature (see for example Blanco, Brennan and Marsh (2005)). I report in Table 2 some descriptive statistics on the basis of the 14 dealers in my dataset\(^\text{23}\). In Figure 4, I plot the approximated yield spread of a 5-year bond (thick line) and the CDS spread (thin line) for some of the banks. As the Table and the Figure show, the basis is usually negative, because the CDS spread is lower than the corresponding bond yield spread.

Three important facts emerge from Figure 4. First, the basis has greatly increased in absolute value for most banks during the crisis, even if it was not zero before (reflecting the cost of capital and other liquidity effects on bond prices as discussed in Section 4). Second, movements in the basis are correlated

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\(^{20}\)In fact, for Credit Suisse I use the series from Bloomberg because that from Markit is incomplete

\(^{21}\)Note that after September 15 2008 both Lehman Brothers and Merrill Lynch drop out of the group. Lehman brothers went bankrupt on that date. Merrill Lynch was acquired by Bank of America, which is in the group of 14 banks considered. Therefore, in order to avoid double-counting the same institution, I drop Merrill Lynch on that date.

\(^{22}\)See Duffie (1999)

\(^{23}\)Because of the unavailability of 5-year maturity bonds at all times, the approximate 5-year bond spread is obtained by linearly interpolating the yield spreads of bonds of maturity closest to 5 years.

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32
across dealers, but there is heterogeneity especially during the recent crisis. This in turn results in a higher degree of asymmetry across banks in the marginal and pairwise default probabilities implied by bond and CDS prices. Finally, Figure 4 and Table 2 show that in some cases the basis drops to zero - implying that counterparty risk for that institution is zero. In some cases, the basis even becomes positive (which again will be interpreted as zero counterparty risk). This can be due to noise in bond pricing data. It can also occur when in response to bad news CDS spreads jump upwards quickly while bond yields react with a lag, because of the lower liquidity of the bond market. Of course, one limitation of the methodology presented in this paper is that all such events will be interpreted as a drop in counterparty risk, at least until the bond yields adjust. However, since this phenomenon typically occurs for a few days, it will not affect the general behavior of the bounds. Smoothing the bounds will further reduce the problem.

6 Results

This section presents the results of the estimation of the bounds. In the first part, I analyse the results about the time series of systemic risk during the financial crisis, under different sets of assumptions, and comparing bounds obtained using the full information set to bounds obtained under interesting alternative information sets. The second part looks at the configuration of the network at the upper bound and studies the individual contributions to systemic risk.

6.1 Systemic risk

6.1.1 Bounds with no restrictions on liquidity

I begin the analysis imposing no restrictions on the liquidity process for bonds, i.e. using the constraints

\[ P(A_i) \leq h_i(0) \]  

in computing the bounds. As explained in Section 4, this imposes the least binding restrictions on the implied risk-neutral marginal probabilities of default. They could be as low as zero (the whole bond/CDS basis is explained by liquidity), or as high as \( h_i(0) \), the probability obtained from bond prices under the assumption that the liquidity process \( \gamma_i^t \) is equal to zero. At the same time, the constraints derived from CDS spreads implicitly impose a lower bound on the marginal default probabilities. This is due to the fact that counterparty risk cannot be negative, and since

\[
\left[ P(A_i) - (1 - S) \left( \frac{1}{N-1} \sum_{i \neq j} P(A_i \cap A_j) \right) \right] \frac{T \delta(0,T)(1 - R)}{\sum_{s=1}^{T} \delta(0,s-1)} = \bar{z}_i
\]
we must have
\[ P(A_i) \frac{T \delta(0, T)(1 - R)}{\sum_{s=1}^{T} \delta(0, s - 1)} \geq z_i \]

Figure 5 plots the bounds for the average monthly probability of default of at least one \( (P_1, \text{first panel}) \) to at least four \( (P_4, \text{fourth panel}) \) institutions. To make the graphs more readable, I plot a 3-day moving average of each bound; the same is true for Figures 6–8. The thin line represents the upper and lower bounds obtained using only bond prices, i.e. imposing only constraints (7) but not constraints (8) in the maximization (minimization) problem. The dotted line, on the contrary, imposes only constraints (8) but not constraints (7): these bounds are obtained using only CDS data. Therefore, both these sets of bounds ignore the information on counterparty risk contained in the bond/CDS basis. The thick line represents the bounds obtained using all available information: the full-information bounds. They are the optimal bounds conditional on the prices of bonds and CDSs that we observe.

Under these minimal assumptions on liquidity premia, all bounds indicate an increase in the maximum possible amount of systemic risk during the financial crisis, until early 2009, followed by a decrease back to the levels of 2007 by the end of 2009. At the same time, the lower bounds for systemic risk are different from zero for \( P_1 \), the probability of at least one bank failing, but are zero for all \( P_r \), with \( r > 1 \). What emerges from this Figure suggests a significant increase in the probability that at least one bank defaults \( (P_1) \), but does not allow to pin down whether a similar increase occurred for the default risk of more than one institution. However, it suggests that uncertainty over systemic risks increased during this period (consistent with the analysis of Section 3, which relates changes in the low-order probability to changes in the width of the bounds). This means that, if one were to combine the information contained in these securities with information coming from other sources (for example, prior beliefs), there would be little room for disagreement about systemic risk before the crisis. However, during the crisis, and up until 2009, CDS and bond data would not be sufficient to exclude an increasingly pessimistic view of systemic risk.

Comparing the bounds obtained using different information sets, an interesting pattern emerges. The full-information bounds, that use together bond and CDS spreads, are noticeably tighter than the bounds that only use bond prices, but, in general, do not seem to improve much over the CDS-only bounds. The reason for this is that the upper bound on marginal default probability derived from bond prices \( (h_i(0)) \) is often so high that constraint (7) is not binding at the maximum of systemic risk. This result comes from internal consistency of the probability system, that implies that in these cases counterparty risk cannot be so high to explain the whole bond/CDS basis. Since normally the basis is relatively large compared to the CDS spreads, when these constraints are not binding the full-information bounds do not improve over the CDS-only bounds (see Section 7 for a more detailed discussion of this mechanism).

However, if in some periods the bond and CDS spreads increase enough while the basis does not increase as much, the constraints imposed by bond prices start to bind. This is due to the fact that the basis is now small relative to the CDS and bond spreads. The full-information bounds capture this information and allow to impose a tighter upper bound to systemic risk: the bounds are particularly informative in these
cases. This is precisely what happens during the peak periods of the crisis (Bear Stearns and Lehman), in which the probability of \( r > 1 \) institutions defaulting is low according to the full-information bounds but high according to the other, simpler, bounds.

Not only do the optimal bounds become tighter during the key episodes of 2008: they also allow an interesting decomposition of the movements in bond yields and CDS spreads into idiosyncratic and systemic default risk. This decomposition emerges clearly if we compare the top panel to the bottom three panels of Figure 5. The top panel indicates that the probability that at least one bank would default spiked during these episodes: both the upper and the lower bound are very high in this period, and the bounds are tight. In the same periods, the upper and lower bounds for the probability of at least two, three and four banks defaulting do not show similar spikes: systemic risk as measured by \( P_2, P_3 \) and \( P_4 \) is shown to be low. This tells us that during these episodes what spiked was mostly idiosyncratic risk, while a similar increase in the default risk of more than one institution did not occur.

Note that we can’t reach similar conclusions if we look at the bond-only or CDS-only bounds. For these bounds, the upper bound on systemic risk increases in a similar fashion in the four panels, and the lower bound stays at very low levels (or 0, for \( r > 1 \)) for the whole sample. Because of that, these bounds do not allow to capture the difference between idiosyncratic and systematic risk. Looking at simple measures like those plotted in Figure 1 would in fact lead to opposite conclusions - that systemic risk spiked during these episodes.

Looking at Figure 5, one might be tempted to say that systemic risk decreased in the Bear Stearns and Lehman episodes: the lower bound for systemic risk stays at zero while the upper bound tightens. This, however, is not the correct interpretation. The bounds simply reveal that systemic risk was low during these times, but they do not establish that it was higher before or after. As discussed above, apart from these periods the full-information bounds are not tighter than the CDS-only bounds, because the bond/CDS basis is too wide to be informative about counterparty risk. Therefore, for times before and after the collapses of Lehman Brothers and Bear Stearns, using the full-information bounds does not allow to distinguish between idiosyncratic and systemic risk. In the next Section I will show how imposing stricter, yet plausible, lower bounds on liquidity allows a further decomposition of systemic risk and idiosyncratic risk that cannot be achieved under the simple assumptions that \( \gamma_i^t \geq 0 \): the full-information bounds will be more informative than the CDS-only bounds even before and after the Bear Stearns and Lehman episodes.

### 6.1.2 Bounds with assumptions on liquidity

As explained in Section 4, if we assume that the liquidity cost of holding a bond is at least \( \gamma_i^t \), the upper bound on each of the marginal default probabilities decreases to \( h_i(\gamma_i^t) \). In turn, this means that the bounds on systemic risk become tighter. In addition, the effect of liquidity is different across dealers at each point in time, and therefore tighter assumptions on liquidity increase the asymmetry of the network. For banks whose yield spread is just slightly above the CDS spread, a small increase in \( \gamma_i^t \) implies that
the counterparty risk priced in the CDS contract on $i$ decreases proportionally by a large amount. For banks for which the difference is very high, the same increase in $\gamma^i_t$ will have little effect on the amount of counterparty risk compatible with CDS spreads. In turn, as discussed in Section 3, a higher asymmetry of the network implies tighter bounds. For similar reasons, the effect of liquidity assumptions on the measure of systemic risk varies over time.

Following the analysis presented in Section 4, I first calibrate $\gamma^i_t$ for each bank so that the average difference between the yield spread and the CDS spread (the bond/CDS basis) during 2004 is attributed to the liquidity premium. This implies a liquidity component of the yield spread of about 50-60% in 2004 and 2004. This is in line with the calibrations of Huang and Huang (2003) and Longstaff, Mithal and Neis (2005).

The top and middle panels of Figure 6 report the bounds on the monthly probability of at least four institutions defaulting ($P_4$) when $\gamma^i_t = 0$ (top panel), and when $\gamma^i_t$ is calibrated to match the bond/CDS basis of each bank in 2004 (middle panel). Lowering the upper bound on the marginal probabilities of default has three effects on the measures of systemic risk. First, it directly rules out some high values for the default probabilities of each bank. Second, it indirectly lowers the maximum amount of counterparty risk present in CDS contracts by lowering the basis for all banks. Third, to the extent that the effect is different across dealers, it increases the asymmetry of the network and therefore the gain in using all the information available.

The bounds constructed using bond prices but not CDS spreads reflect only the first effect. As tighter assumptions on liquidity are imposed, the upper bound shifts down relatively uniformly. The bounds that use only CDS data, instead, are not affected at all by bond liquidity assumptions. Instead, the optimal bounds reflect all three effects, and therefore become significantly tighter than both other sets of bounds. The message that during the peak episodes of 2008 systemic risk did not spike is reinforced under these assumptions.

The bottom panel of Figure 6 plots the bounds obtained by calibrating the liquidity process to the one that explains the basis of nonfinancial firms, as explained in Section 4. In particular, I look at the nonfinancial firms that compose the CDX IG index (a standard index of CDS spreads on investment-grade bonds), restricting the search to those with credit rating of A1 or higher (according to Moody’s). For 8 of these banks I have enough data to include them in the estimation of $\lambda^*_t$. This lower bound on $\gamma^i_t$ starts at the average level of 2004, stays approximately constant until 2007, and during the crisis it increases by up to three times.

Using a time-varying lower bound for liquidity strengthens the results obtained so far. Systemic risk is confirmed to be low until late 2008. From the second half of 2008, the upper bound on systemic risk starts increasing significantly - with no spikes either around the Bear Stearns or the Lehman Brothers collapses. In addition, it is now easier to see that systemic risk did not really decrease in early 2008, but was generally low until the second half of 2008. The same cannot be said for the year following Lehman’s default: bond and CDS prices are consistent with an increase in systemic risk.
The story Figure 6 tells is then in contrast with several accounts of the crisis that underlined the fear of a systemic event even before 2009. In a sense, the bounds point out an internal inconsistency of many of these accounts. High systemic risk in the financial sector requires low CDS spreads (relative to bond yields) for contracts written by large dealers against the default of other dealers. Since the counterparties to those contracts are in fact the same dealers that are at the core of the financial network, a high risk of systemic default ought to reduce the value of insurance contracts bought from them. However, during some of these episodes, the bond/CDS basis was not wide enough to correspond to such systemic risk. This effect is even stronger when part of that basis is accounted for by liquidity premia in the bond market.

6.1.3 Full-information bounds and average-information bounds

An interesting question is whether one could achieve similar results on systemic risk by looking only at the average bond and CDS spreads rather than using the full information set to bound systemic risk. As shown in Section 3, bounds obtained using only average information coincide with bounds obtained in a fully symmetric system. Therefore, the difference between the two sets of bounds highlights the role of asymmetries in the spreads and in the bond/CDS basis for learning about systemic risk.

Figure 7 plots the bounds obtained under the full and the average information set, and under the weakest liquidity assumption \( \gamma_i^t \geq 0 \). The average information set is obtained by replacing the two sets of constraints (3) and (4) with one constraint each, obtained averaging each of the two sets across \( i \)'s. The Figure shows that looking at average spreads only allows us to distinguish between idiosyncratic risk and systemic risk in some cases (Bear Stearns) but not in others (Lehman). The reason is that in the first episode the spreads and bases of most banks moved in a similar way, but in the second case the risks were concentrated mostly in some banks: in this case, the asymmetry of the network is crucial in learning about systemic risk.

Figure 8 shows the bounds on \( P_r \) for the different assumptions on \( \gamma_i^t \) discussed in Section 4. It shows that liquidity assumptions tighten both the full information bounds and the average information bounds, but the effect is stronger for the former. In addition, the idiosyncratic/systemic risk decomposition of the months of September-November 2008 is only achieved by the full-information bounds, under all assumptions on liquidity.

6.2 Individual contributions to systemic risk

The method described in this paper to obtain the bounds on systemic risk also allows to study the evolution of the default risk of each bank and its relation with the rest of the network. In particular, by solving for the probability system that attains the upper bound, we can learn about the marginal and pairwise probabilities of default in the scenario of highest systemic risk. In this section I look at the bounds for \( P_4 \) obtained calibrating the liquidity process to the levels of 2004 (as in the middle panel of Figure 6). In addition, we can also study the contribution of each bank to systemic risk.
We can start by looking at the configuration of the financial network at the upper bound of systemic risk in terms of marginal and pairwise probabilities. Figure 9, for example, plots a snapshot of part of the network as of 08/04/2008, more than a month before Lehman’s collapse. The nodes of the diagram represent the individual banks and are associated with monthly marginal probabilities of default. The segments that connect the nodes report the joint default probability of the two intermediaries. The graph reflects the market beliefs about the risks of default of these banks individually and in pairs - in the most interconnected configuration.

In decreasing order of marginal default risk we find Lehman Brothers, Merrill Lynch, Morgan Stanley, Citigroup, and Deutsche Bank. In addition, we can see that the pair at highest risk of default is Merrill Lynch with Lehman Brothers, followed by the pair Merrill Lynch and Citigroup. It is evident from this graph how the prices of bonds and CDSs were consistent with a high joint default risk of the first two banks a while before the weekend in which both underwent severe problems. The rest of the graph shows considerable heterogeneity in the marginal probabilities of default, but especially so for the pairwise probabilities. Citigroup appears to be more connected to Lehman and Merrill than Morgan Stanley, even though it has a lower marginal default probability than the latter. Deutsche Bank, whose basis is relatively small, displays a low marginal probability of default and especially low joint default probabilities with the other banks plotted.

Using a similar approach, for each pair of banks $i$ and $j$ we can track the evolution of $P(A_i), P(A_j)$ and $P(A_i \cap A_j)$ over time. Figure 10 plots a weekly moving average of these probabilities for three different pairs (all combinations of Lehman, Merrill Lynch, and Citigroup). The upper panel reports the marginal probabilities, and the lower panel reports the joint probabilities. These graphs confirm the relatively high degree of heterogeneity and variability in marginal default probabilities across banks, but even more for joint probabilities. Particularly interesting is the fact that, for the two months preceding the collapse of Lehman brother and the buyout of Merrill Lynch, markets were anticipating (in the worst case scenario) high joint default probabilities for the two even though the marginal probabilities of each of them did not increase as much. In other words, the information in bond and CDS spreads would not be enough to rule out a relatively high probability that Lehman and Merrill Lynch could default together - while it would be enough to bound this risk to much lower levels for all other pairs of banks, as well as jointly for several banks.

By finding the probability system that attains the bounds we can also study the contribution of each institution to systemic risk and track its evolution over time. One way to capture this is to compute the probability that institution $i$ is involved in a multiple default event:

$$Pr\{\text{at least 4 default } \cap i \text{ defaults}\}$$

Figure 11 plots this contribution at the upper bound for four banks (Citigroup, Lehman Brothers, Merrill Lynch and Bank of America) as well as the average across the other 10 banks (to improve readability, I plot again a weekly moving average). Some interesting results emerge from this graph. First of all, there

38
is again heterogeneity across firms, both in the levels and in the changes. While the contribution to systematic risk increases for all banks after August 2007, the growth is faster for Lehman and Merrill Lynch than for the other banks: markets seem to have anticipated the increased systemic importance of these two institutions before the weekend of September 15. After September 2008, the contribution of different banks seems to spike in different periods. For example, Citigroup’s contribution seems much larger than the other banks in late 2008 and early 2009, and then reverts to the average in the second half of 2009.

Figures 9 to 11 also allow us to shed some additional light on the mechanics of the bounds on systemic events reported in Figures 5-8. Looking at those Figures, we concluded that in some of the key episodes (particularly Bear Stears and Lehman) idiosyncratic risk spiked while systemic risk remained relatively low. These additional figures show us the underlying source of the result. Even in the scenario of highest systematic risk, for most banks the contribution to systemic risk is relatively small. Only few banks, among which Lehman and Merrill Lynch, were perceived to be at a high risk of default. An increase in their idiosyncratic risk of default went together with an increased risk of joint default with a few other institutions. Since these spikes in default risk involved a few banks at a time, they did not directly result in a large generalized increase in systemic risk across the financial network. At the same time, the increase in correlated risk for some dealers also made the network more asymmetric. This explains the gains in terms of width of the bounds when using all the information available.

7 Robustness

In this section I study the robustness of the main results of the paper to different assumptions. I first examine robustness to assumptions about the recovery rate of bonds ($R$) and CDSs ($S$). Then, I estimate the model using different assumptions about the process for the hazard rate of default, and compare the results with those obtained in Section 6. Finally, I discuss the underlying assumptions needed to perform my analysis using American and European banks together, and study the robustness to the exclusion of European banks.

7.1 Recovery rates

7.1.1 Assumptions on the (constant) recovery rates

As shown in equations (1) and (2), the expected recovery rate of bonds in case of default, $R$, affects the prices of both bonds and CDSs. A higher expected recovery rate in case of default increases the value of a bond, and at the same time decreases the value of CDS insurance written on that bond, since the payment from the CDS sellers covers only the amount of bond value not recovered in default. The effect of an increase in $R$ then is to decrease both the yield spread of the bond and the corresponding CDS spread, with a theoretically small effect on the bond/CDS basis. Since the measure of systemic risk constructed in this paper depends directly on the basis, we would expect the bounds on systemic risk to be relatively...
robust to changes in $R$.

The case for the recovery rate of CDSs when double default occurs, $S$, is more interesting. As evident from equation (2), for given marginal probability of default $P(A_i)$, when $S$ is higher a higher joint default risk is required to match the CDS spread. When $S$ is low, even a small amount of joint default risk will push the CDS spread down, so that the entire bond/CDS basis is explained by double default risk. An increase in $S$ allows for higher counterparty risk to explain the basis. This effect therefore would imply that the upper bound on systemic risk should increase with $S$.

However, there is another, opposite, force at play. If $S$ increases enough, and becomes high relative to the basis, it might happen that the amount of counterparty risk required to explain the basis is so high that it would not be consistent with the marginal default probabilities of the single banks. The internal consistency requirements of the probability system prevent that a really low CDS spread could be explained by a high marginal default probability of the reference entity and an even greater amount of counterparty risk. This happens because joint default risk, $P(A_i \cap A_j)$, is bounded above by the marginal default risk itself $P(A_i)$, and, as long as $S < 1$, the CDS pricing equation imposes that an increase in $P(A_i)$ must be matched more than one-for-one by an increase in $P(A_i \cap A_j)$ - therefore, at some point the joint default risk will attain its upper bound, $P(A_i)$. At this point, further increases in $S$ actually decrease the maximum possible amount of systemic risk, by letting a greater fraction of the basis be explained by liquidity. This force then pushes in the opposite direction: as $S$ increases, systemic risk has to decrease.

In the Appendix, I show these results formally in the symmetric case. In the asymmetric case, the two opposite forces counterbalance each other in such a way that the bounds on systemic risk are in fact not very sensitive to assumptions on $S$. The reason is that in an asymmetric network different banks will have different bases, so that for most values of $S$ the two effects described above will operate for some banks in the one direction and for other banks in the opposite direction, counterbalancing each other.

Table 3 reports the results of the estimation of the upper bound on systemic risk under different assumptions for $R$ and $S$. $R$ varies from 0% to 40%, and $S$ varies from $R$ to 100% (it cannot be smaller than $R$ because CDSs are of equal seniority, and are partly collateralized). The bounds are obtained under the assumption $\gamma_i^l \geq 0$, so they correspond to those plotted in Figure 5.

For each combination of $R$ and $S$, columns 3 to 6 report the average ratio of the full-information upper bound to the bonds-only and the CDS-only upper bound. For example, column 3 reports

$$\frac{1}{T} \sum_{t=1}^{T} \left( \frac{P_{full}^{1t}}{P_{bond-only}^{1t}} \right)$$

while column 4 reports the same statistic using $P_A$ instead of $P_1$. These columns tell us how informative the full-information bounds are relative to the other bounds. In the baseline case, the full-information bounds are on average about 50% tighter than the bonds-only bounds. However, as visible from Figure 5, they improve on the CDS-only bounds only for $P_A$, and even then, the bounds are on average 95% of the CDS-only bounds. These results are very robust to assumptions on $R$, as well as $S$. The only
exception happens at extremely low values of $S$, where a very large improvement is seen even relative to the CDS-only bounds.

While the optimal bounds improve on CDS-only bounds just slightly, on average, from Figure 5 and 6 we know that they improve noticeably during some periods, specifically around Bear Stearns’s collapse and following Lehman Brothers’s bankruptcy. Columns 7 to 10 report, for these two episodes, a statistic that summarizes the distinction between idiosyncratic and systemic default risk. For each episode (call the respective time periods $T_{ep} \equiv (t_{1ep}, t_{2ep})$), I report the ratio

$$\frac{1}{t_{2ep} - t_{1ep}} \sum_{t \in T_{ep}} \left( \frac{P_{4full, t}}{P_{1full, t}} \right) \left( \frac{P_{bond-only, t}}{P_{bond-only, t}} \right)$$

at the upper bounds. This number captures the ratio of the upper bound on $P_4$ and $P_1$ for the full-information bounds, scaled by the same ratio for the partial-information bounds (the Table also reports the ratio obtained scaling by CDS-only bounds). Remember from Figure 5 that while for the bond-only and CDS-only bounds both $P_4$ and $P_1$ showed similar-shaped spikes around March and September 2008, the full-information bounds were lower for $P_4$ than they were for $P_1$. The statistics presented here allow to capture this information in one number. When the statistic is 1, these ratios are the same using full information or just the partial information sets. Because we know that the partial information sets (bonds-only and CDS-only) do not allow a decomposition between idiosyncratic and systemic default risk, when the number is 1 the full-information bounds are not allowing such a decomposition either. When the number is lower than 1, the probability that at least 4 institutions default is proportionally lower than the probability that at least 1 defaults according to the full-information bounds, therefore achieving a decomposition of systemic and idiosyncratic risk.

In the baseline case of $R = 0.3$ and $S = 0.3$, just as Figure 5 showed, this decomposition is clear for both episodes. As Table 3 shows, the results are very robust to assumptions on $R$ and $S$, even for $S$ close to 1. As we would expect, the results disappear if we approach $S = 1$, because in this case the information coming from bond prices is not used in the full-information bounds, so that the latter coincide with the CDS-only bounds.

Table 4 explores how these results change under different liquidity assumptions (for reasons of space I only report $R = 0.3$ and 0.4). When liquidity assumptions are introduced, they shift downwards the bond-only bounds and the full-information bounds, closer to the CDS-only bounds. The first four columns of Table 4 show that the gain of using full-information bounds is still strong relative to bonds-only bounds, and becomes much stronger relative to CDS-only bounds. This is robust to assumptions on $R$ and $S$. The last four columns show, as expected, a general improvement in the decomposition between systemic and idiosyncratic risk during the Bear Stearns and Lehman episodes, for all values of $R$ and $S$.

\[^{24}\text{For Bear Stearns I include the month before and the month after March 15; for Lehman I include the month after September 15. The choice is made in order to capture the main results of Figure 5.}\]
To sum up, the main results of the paper - that the full-information bounds improve significantly over the other bounds, and that they allow a decomposition of systemic and idiosyncratic risk especially during some key episodes of the crisis - are very robust to assumptions on $R$ and $S$. The results disappear only if the recovery rate of CDSs is assumed to be very close to 100% even in case of double default, arguably an unrealistic case given the collateral limitations discussed in Section 2 and, more in detail, in Appendix A.

7.1.2 Time varying recovery rates

Above I have studied robustness to different assumptions about $S$ and $R$, when these are assumed to be constant during the whole sample period. In theory, it is possible that these recovery rates vary over time in a way that affects the results on the time-series of systemic risk presented in Section 5. Suppose in particular that at every time $t$ bonds and CDSs are still priced assuming that at all future periods $t+s$ the recovery rates are constant and equal to $S_t$ and $R_t$; however, let now $S_t$ and $R_t$ vary over time. How will this affect the bounds?

As explained above, changes in $R$ have a small effect on the bond/CDS basis because they move the yield spread and the CDS spread in the same direction. Time variation in $R$ therefore will likely have minor effects on the bounds. A more relevant concern is the case of reduction in the recovery rate of CDSs, $S$, in times when systemic risk increases. However, it is easy to see that this case actually reinforces the main results (low systemic risk before mid 2008 and decomposition of idiosyncratic and systemic risk in 2008): if the recovery rate $S$ becomes smaller during the key episodes of the crisis, then joint default risk (and with it systemic risk) has to be be smaller as well. This stems once more from the fact that during these episodes the bond/CDS basis is small relative to CDS and yield spreads. When $S$ is reduced, the probability of double default has a greater effect on the basis. To still match the basis, joint default risk has to decrease. Therefore, the main results in the paper will be robust to a decrease in the recovery rate $S$ in times of crisis.

7.1.3 Stochastic recovery rate on bonds $R$

Another possibility is that when pricing bonds and CDSs, agents incorporate the possibility that recovery rates might be stochastic and correlated with the default events in the financial sector. In particular, one could think that recovery rates of both bonds and CDSs might deteriorate the more defaults happen in the financial system.

Because of the limited data available, it is difficult to solve explicitly for the case of stochastic recovery rate. However, it is possible to show that under simple modeling assumptions, the effect of a stochastic recovery rate on the bounds on systemic risk can be decomposed in two steps. First, starting from the case of constant recovery rate, we shift this recovery rate $R$ downwards for both bonds and CDS. The previous robustness tests have shown that the results are robust with respect to this change. Second, the present value of bonds is increased by a certain amount - and the present value of payment from the CDS
contract is decreased by approximately the same amount. This implies that the yield spread and the CDS spread shift down by a similar amount, so that the basis again remains approximately unchanged. Both these effects suggest that the effect of a stochastic recovery rate on the bounds on systemic risk is not of first-order importance.

These results, reported in the Appendix, can be obtained under the assumption that the recovery rate on bonds is $R_H$ whenever one bank defaults alone, and $R_L < R_H$ whenever two or more banks default.

### 7.2 Assumptions about the pricing model

While the procedure described in this paper to construct bounds on systemic risk imposes no assumptions on the structure of higher-order correlations of default risks, it still imposes a pricing model in order to extract marginal and joint default probabilities from bond and CDS prices. In particular, these securities are priced at each time $t$ assuming a constant monthly default hazard rate from time $t$ to maturity. This section relaxes this assumption by allowing a more flexible form for the hazard rate process. In particular, for each institution $i$, from the perspective of an agent pricing bonds at time $t$, the hazard rate at time $t + s$ follows the deterministic autoregressive form

$$h_{t+s} = (1 - \rho_t)\bar{h}_t + \rho_t h_{t+s-1}$$

where parameters $h_t$, $\bar{h}_t$ and $\rho_t$ are determined at time $t$. As before, all bonds are priced at every time $t$ assuming that the hazard rate process is known for all future dates.

At each time $t$, then, I can use bond prices to estimate $h_t$, $\rho_t$ and $\bar{h}_t$. This representation allows to capture cases in which the hazard rate is higher at shorter horizons and then reverts to a lower long-term value. I can therefore construct the bounds on systemic risk using $h_t$, the probability of default in the month after $t$.

As discussed in Section 4, the main problem with this approach is that while it is easy to estimate a more flexible function for the marginal hazard rate of default using bond prices, CDS data do not contain enough information to estimate a similarly flexible process for joint default risk (because at each time $t$ we only observe the spread of one CDS, with maturity of 5 years). To tackle this limitation, I assume that the joint hazard process replicates the shape of the marginal hazard process of the reference entity: the process decays at the same rate ($\rho_t$) and displays the same ratio between short-term and long-term default hazards ($h_t/\bar{h}_t$). The details of the estimation method and of the additional assumptions involved are reported in the Appendix.

Figure 12 plots the results for $P_1$ and $P_4$ under the three different liquidity assumptions discussed in Section 4. With a more flexible hazard process, during peak periods idiosyncratic default risk increases noticeably. The graphs confirm the main results of the paper, concerning the level of the full-information bounds relative to the CDS-only and bonds-only bounds, as well as the distinction of idiosyncratic and systemic risk in 2008.

43
7.3 Assumptions about the exchange rate

The construction of the bounds on systemic risk involves the estimation of risk-neutral probabilities from bond prices and of joint default probabilities from CDS spreads. Using probabilities obtained from different securities to obtain risk-neutral probabilities of joint default requires additional assumptions if the securities are denominated in different currencies. In particular, while most bonds issued by American firms and the CDSs written on them are denominated in dollars, European firms issue several bonds in Euros and in other currencies, and the CDSs written on them are denominated in Euros.

To simplify the discussion, consider one-period bonds issued by bank \( i \). Call \( m_{se} \) the stochastic discount factor of a US investor in state \( (s, e) \) where \( s \) indicates the default state of bank \( i \) (\( s = i \) or 0) and \( e \) is the exchange rate with a foreign currency. Call \( f(s, e) \) the joint density function of \( s \) and \( e \). Decompose \( f(s, e) \) in the default probability and the conditional density of the exchange rate:

\[
f(s, e) = \pi_s f_s(e)
\]

so that \( \pi_s E[m_{se}|s] \) is the price of a security that pays 1 if default state \( s \) happens. The price of a state-contingent security that pays a unit of foreign currency if default state \( s \) happens is then \( \pi_s E[e m_{se}|s] \).

The Appendix shows that a sufficient condition for estimating risk-neutral default probabilities using bonds denominated in different currencies (using the risk-free rates denominated in the respective currencies to discount cash flows) is:

\[
\frac{\pi_s E[e \cdot m_{se}|s]}{\pi_s E[m_{se}|s]} = \frac{E[e \cdot m_{se}]}{E[m_{se}]} \tag{9}
\]

which requires that the relative price of domestic and foreign risk-free securities is the same as the relative price of domestic and foreign state-contingent securities that pay off if \( i \) defaults.

Similarly, consider the case of a CDS written by one bank on the default of another bank. The CDS is written on a european bank \( (i) \) but the counterparty \( (j) \) is american. The contract is denominated in euros. In this case, let \( s \) represent the combination of default states of the two banks, so that it can take values \( i \) (only \( i \) defaults), \( j \) (only \( j \) defaults), \( ij \) (both default), and 0 (none defaults). Again, as long as the european yield curve is used to discount cash flows for euro-denominated CDSs, a sufficient condition to estimate correct risk-neutral probabilities is equation (9), if it holds for every value of \( s \).

Therefore, a sufficient condition for the validity of the bounds on systemic risk presented in this paper is that equation (9) holds for all possible default states of all banks. Of course, it is reasonable to assume that the relative price of dollar-denominated and foreign currency-denominated default-contingent securities might be different depending on the default state (think for example of a flight-to-quality to US securities if several banks default). As a robustness test for the validity of the bounds in case these conditions are violated, I perform the estimation exercise including only American firms, for which all bonds and CDSs are dollar-denominated. Figure 13 shows that the main results from Figures 5-8 still
hold in this case.

8 Conclusion

This paper shows that bond prices and CDS spreads represent a rich information set for learning about joint probabilities of default of intermediaries in the financial network. This information set can be used to construct bounds on the probability of systemic events, defined as the probability of several institutions defaulting within a certain time horizon. These bounds are the best possible ones given the available information, and they are obtained as the solution to a linear programming problem.

Even without imposing assumptions on the liquidity premia in bond markets, the bounds obtained in this way are significantly tighter than the ones obtained from bond prices alone. They are also tighter than bounds that only use CDS spreads, but only in times in which CDS and bond spreads increase while the bond/CDS basis remains relatively small. Adding relatively mild assumptions on the liquidity process that affects bond prices allows to further tighten the bounds. According to these bounds, systemic risk started to increase in August 2007 but remained relatively low until mid-2008. After that, the bounds widen considerably, which would be consistent both with an increase in systemic risk and with an increase in uncertainty about the severity of such risks. Finally, the bounds converge to lower values starting in the second half of 2009.

The paper shows how it is possible to decompose the changes in bond and CDS spreads across the network into an idiosyncratic and a systemic component of default risk. By using the full information set available from observed prices, the bounds show that some of the spikes in the prices of bonds and CDSs of financial institutions during the crisis correspond to increases idiosyncratic default risk (that affected a small number of banks) more than systemic risk. This decomposition is important to understand the markets’ perceptions of risk during the financial crisis.

The method described can also be used to track the contribution to systemic risk of the institutions in the financial network, as well as to obtain a representation of the network in the scenario of highest systemic risk. An interesting result that emerges from this analysis is that markets seem to have anticipated the possible systemic nature of the risk exposure of some banks (Lehman and Merrill Lynch), as well as the links between them, a while before these institutions faced a severe crisis in September 2008: the upper bound on the joint default probability and the contributions to systemic risk of these two banks increased dramatically already in July 2008.

The approach used in this paper has several limitations. First, it is capturing market perceptions about probabilities rather than the true probabilities. This means that just as securities can be mispriced, the bounds can reflect various imperfections and mispricing that happen in financial markets, including slow incorporation of information and underestimation of risks. In addition, because they are reduced-form, they do not allow to distinguish whether the low systemic risk estimated, for example, around March 2008 was due to the configuration of structural links between the banks or because of the expectation of
government intervention in case multiple banks defaulted (which, after all, is what happened in September 2008). On the other hand, the bounds do allow to rule out some of the accounts about perceived risks, such as the fear of the government leaving several banks to default in case shocks propagated throughout the system.

Second, while the approach is built to minimize the number of assumptions about the correlation structure of the network, especially regarding high-order joint risks, some assumptions are still necessary to obtain the bounds. For example, estimating risk-neutral marginal and pairwise default probabilities from prices requires imposing a pricing model and taking a stand on liquidity in bond markets. Some of these issues are tackled in the paper, by calibrating liquidity and performing robustness tests on the pricing model - however, further modifications in the set of assumptions may lead to different results. Finally, since it is difficult to distinguish the effect of counterparty risk on the bond/CDS basis from other factors that may affect it, it is impossible to completely rule out alternative explanations of the basis, that attribute a less important role to counterparty risk. These alternative explanations might have different implications for the measurement of systemic risk.

However, to the extent that we believe counterparty risk plays a role in determining CDS spreads, the method presented in this paper can help understand the evolution of systemic risk in the financial markets during the recent crisis. Because of the possibility of constructing the bounds in real time, this measure could also be used to complement other measures in monitoring the market perceptions of systemic risk.
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Appendix A - Counterparty risk and collateral in CDSs

Given that the incorporation of counterparty risk and collateral in the pricing of Credit Default Swaps is central for this study, it is worth discussing the related assumptions in detail, especially in reference to the observed events related to the credit crisis of 2008-2009. An initial question is whether counterparty risk was perceived at all by market participants. The growth of the percentage of OTC derivative contracts covered by some form of collateral (ISDA) confirms this indirectly. Besides, documents from practitioners confirm that the issue was considered by financial participants throughout the crisis (for example, see Barclays Capital (2008)).

A second issue involves the extent to which, given that agents were aware of counterparty risk, this risk was priced in the CDS spreads; after all, collateral might have been sufficient to eliminate most of it. There are several reasons to believe that this is not the case. First, we know from ISDA that only about 2/3 of the contracts were covered by collateral agreements. Besides, calculations by Singh and Aitken (2009) and Singh (2010) show that, at the end of 2009, large financial institutions still carried large under-collateralized derivative liabilities. In particular, they compute the total value of “residual derivative payables” - liabilities from derivative positions after netting under master netting agreements and in excess of the collateral posted. For the 5 largest US dealers this amount was more than $250bn. Even though these numbers include all derivative contracts, and not only CDSs, they suggest a general under-collateralization of derivative positions from these counterparties. Second, it is also known that most collateral requirements were imposed on hedge funds and smaller dealers, while large intermediaries often had much less stringent collateral requirements, if any at all. Third, the amount of collateral required to be posted (collateral calls) depends on the credit risk of the reference entity and of the protection seller. Because of this, a high recovery rate in case of double default can be achieved only if both the risks of the counterparty (usually measured by the credit rating) and of the reference entity (as measured by the CDS spread) increase enough before the double default.

While we did see credit downgrades and gradual increases in CDS spreads of banks during the crisis, a crucial episode - the Lehman bankruptcy - shows that recovery rates in the double default cases can indeed be very small. Just before the weekend of the 13th and 14th of September 2008, many institutions were considered at risk (Lehman among them), but neither the credit ratings nor the CDS spreads indicated an extremely high likelihood of immediate default. For example, the Lehman 5 year CDS was trading at around 700bp per year, Merrill’s at 500bp, and the credit ratings of their debt had were still as high as 4 months before, with an implied default probability of less than 0.25% per annum. When Lehman went bankrupt, all CDS contracts written by Lehman were terminated. Its counterparties used their right to seize the collateral and they ended up incurring only small losses. 

At a first glance, it appears that counterparty risk proved to be very small during this episode. However, this is due to the fact that thanks to the intervention of Bank of America and of the government (that saved Merrill Lynch on September 14th and AIG on September 16th, respectively), a double default event never occurred. Consequently, the CDSs written by Lehman and owned by its counterparties were little in the money, because the value of the claims against Lehman was small. The collateral was not even enough to cover the full replacement cost of the contracts when no double default occurred; clearly, it would have been vastly inadequate if another institution, for

25 For credit derivatives, as a percentage of exposure, collateralization went from 39% in 2004 to 58% in 2005, to 66% in 2007 and 2008.
26 Moody’s reports that the claims filed by the other major dealers related to derivative contracts were on the order of 1 to $2bn each.
27 The cost of the AIG contract increased after Lehman declared bankruptcy during September 15th; its CDS spread spiked, but immediately came down the next day, when it was clear that the government would not let AIG default.
example AIG or Merrill Lynch, had been left to fail. Then, the CDS written by Lehman on AIG’s bond would have been in the money for the full amount insured, and the buyers of protection from Lehman would have recovered just a small fraction of their claim by seizing the posted collateral. This example, together with the other salient events of the financial crisis, highlights the limited role of collateral to protect against double defaults which, when they involve financial intermediaries, tend to happen in a sudden and correlated fashion.

Given that collateral is generally not enough to protect against losses in case of double default, we would expect counterparty risk to be priced in the CDS spreads. In particular, the spread on a CDS contract written by a dealer against another dealer’s default should be lower the higher the correlation of the default events of the two. Gathering direct evidence on the extent to which counterparty risk was in fact priced in these contracts is extremely difficult, because its effect on the spread relative to the bond yield can be confused with the effects of other variables, such as liquidity. Arora et al. (2009) document that dealers with high idiosyncratic default risk (as measured by the spread of the CDS written on them) posted quotes systematically higher than the other dealers for the same reference entity, and especially so after Lehman’s bankruptcy. However, they also report that for most reference entities, the difference was quite small, in the order of a few basis points. Note that the study looks at variation of quotes around the daily average, and therefore it filters out all components of counterparty risk which are common to all dealers. Given that the financial crisis affected most financial institutions at the same time, this effect could be quite large. As discussed in Section 4, the bounds I construct are based only on the average quote and therefore do not depend on the cross-sectional variation around it. In fact, on this respect the results obtained in this paper complement those in Arora et al. (2009): their study focuses on the pricing of counterparty risk around the daily average, while this study concentrates on the average level of counterparty risk. Pricing in this paper is obtained allowing (and not imposing) average counterparty risk to explain part of the difference between bond yields and CDS spread. Finally, note that we would expect counterparty risk to have a very small effect for CDSs written most reference entities, because for most firms, especially non-financials, the risk of double default is extremely small. Only for other financial firms, for which double default is a relevant possibility, the theory predicts that counterparty risk could be a first-order component of CDS spreads. Of course, there might be incentives for dealers to post quotes closer to the average of the other dealers: for example, a riskier dealer might prefer posting a quote in line with the other dealers not to signal its weakness. Because the paper focuses on average quotes only, all results hold as long as the average quote posted reflects average counterparty risk across counterparties.

It is also worth mentioning that there are other elements of CDS contracts that potentially affect their spreads and that are not directly addressed in this paper. First, liquidity of the CDS market could influence the CDS spreads, just as bond liquidity is known to affect bond prices. In this paper, I take into account explicitly liquidity premia in bond prices, but not in CDS spreads. For the case of CDSs, liquidity is much less likely to be an issue, especially because they require much less capital at origination and they are not in fixed supply. Also, I abstract from restructuring clauses and the cheapest-to-deliver option sometimes present in CDS contracts. A restructuring clause (under which payment is triggered for simple debt restructuring, in addition to bankruptcy) is more frequent for European bonds, and this results in the contract being triggered in cases close to the Chapter 11 for the US. Berndt, Jarrow and Kang (2006) estimate that the presence of such clause increases the value of the CDSs by 6-8%, and all the results in this paper are robust to an adjustment of CDS spreads of that magnitude. The value of the cheapest-to-deliver option (which allows the buyer to deliver to the seller the cheapest of the defaulted bonds of the same seniority as the reference bond) will be small relative to the CDS spread as long as in default all senior unsecured bonds have similar recovery rates. Additionally, as observed in the Delphi and Calpine defaults in 2005, the high demand for the cheapest bonds might determine shortages of such securities and therefore, anticipating

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28 For an additional discussion of this and on the supporting evidence, see Blanco, Brennan and Marsh (2003,2005).
this, a reduction in the ex-ante value of the option\textsuperscript{29}.

Appendix B - Implementation of the Linear Programming Problem

This appendix describes in detail the algorithm employed to transform the probability bounds problem into a linear programming problem. It also describes the linear approximation to the CDS pricing formula that allows to write the CDS constraints as linear constraints.

B.1 - Linear programming representation in the general case

This section describes the algorithm used to transform the probability problem

\[
\max P_r
\]

s.t.

\[
P(A_i) = a_i
\]

\[
\ldots
\]

\[
P(A_i \cap A_j) = a_{ij}
\]

into the LP representation

\[
\max \, c_r p
\]

s.t.

\[
p \geq 0
\]

\[
i'p = 1
\]

\[
Ap = b
\]

for the general case of \(N\) banks.

Start with a matrix \(B\) of size \((2^N, N)\) whose rows contain the binary representation of all numbers between 0 and \(2^N - 1\). For example, with \(N=4\):

\[
B = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \\
\vdots \\
1 & 1 & 1 & 1
\end{bmatrix}
\]

Note that each row of this matrix corresponds to a particular element of the partition of the sample space described in Proposition 2: the event

\[
A_1^* \cap A_2^* \cap \ldots \cap A_N^*
\]

\textsuperscript{29}De Wit (2006).
where $A_j^* = A_j$ if element $j$ of the row is 1, and $A_j^* = \overline{A_j}$ if element $j$ of the row is 0. The probability system $p$ will then be determined as a vector of $2^N$ elements containing the probability of each of the elements of the partition represented by the $2^N$ rows of the matrix $B$. For example the first element $p_1$ will be the probability of none of the $A$ events occurring, the element $p_2$ will represent the probability that event $A_N$ occurs but none of the other events does, and so on. Finally, the element $p_{2N}$ will represent the probability that all events occur.

The maximization problem presented above tries to find the vector $p$ that maximizes the probability of systemic event of degree $r$ ($P_r$) while satisfying constraints on marginal and pairwise default probabilities, as well as the constraints implied by the definition of a probability measure. The latter are immediate: because the events represented by the rows of $B$ are a partition of the sample space, and $p$ is a probability measure on these events, all elements of $p$ need to be nonnegative and sum to one:

$$p \geq 0$$

$$p' i = 1$$

To obtain in LP form the inequalities and equalities that involve marginal and pairwise default probabilities, note first that because the elements of the partition are disjoint events, the probability of any union of them is equal to the sum of their probabilities. Therefore, to find the probability of an event $A_i$, $P(A_i)$, in terms of $p$, one needs to sum the probabilities of all the elements of the partition in which event $A_i$ occurs. But this is immediate given the representation in $B$:

$$P(A_i) = \sum_{j : B(j,i) = 1} p_j$$

Or:

$$P(A_i) = a^i p$$

for a vector $a_i$ of size $(2^N, 1)$ s.t.:

$$a_j^i = B(j, i)$$

In other words, to find which elementary events form event $A_i$ one needs to find all the rows of $B$ in which element $i$ is equal to 1. The union of these events will coincide with $A_i$, and therefore the sum of their probabilities will be $P(A_i)$. Given the linearity, this sum is equivalent to the product of the vector $p$ with a vector $a_i$, whose elements are ones whenever the corresponding elementary event is a subset of $A_i$.

Similarly, the probability of a joint event:

$$P(A_i \cap A_k) = \sum_{j : B(j,i) = 1 \text{ and } B(j,k) = 1} p_j$$

or:

$$P(A_i \cap A_k) = b^{j^i_k} p$$

for a vector $b_{ik}$ of size $(2^N, 1)$ s.t.:

$$b_{jk}^i = B(j, i)B(j, k)$$

i.e., the probability of the joint default is obtained summing the elements of $p$ s.t. the corresponding element of the partition involves both the occurrence of $A_j$ and of $A_k$. All these constraint can then be collected in the matrix form $Ap = b$. 

54
Finally, the probability that at least \( r \) events occur can be found as follows:

\[
P_r = \sum_{j: \sum_{h=1:N} B(j,h) \geq r} p_j
\]

or:

\[
P_r = c^r' p
\]

for a vector \( c^r \) of size \((2^N, 1)\) s.t.:

\[
c^r_j = I \left[ \sum_{h=1:N} B(j,h) \geq r \right]
\]

where \( I[] \) is the indicator function.

Given this decomposition, the LP representation obtains immediately.

B.2 - CDS pricing approximation

Start from the discretized pricing equation with constant hazard rate and risk-free rate:

\[
\sum_{s=1}^{T} \delta(0,s-1)(1 - P(A_i \cup A_j))^{s-1} z_{ji} = \sum_{s=1}^{T} (1 - P(A_i \cup A_j))^{s-1}(P(A_i) - (1 - S)P(A_i \cap A_j))\delta(0,T)(1 - R)
\]

We can rewrite the equation as:

\[
\frac{z_{ij}}{\delta(0,T)(1 - R)} = \frac{\sum_{s=1}^{T} (1 - P(A_i \cup A_j))^{s-1}(P(A_i) - (1 - S)P(A_i \cap A_j))}{\sum_{s=1}^{T} \delta(0,s-1)(1 - P(A_i \cup A_j))^{s-1}}
\]

and then approximate the right hand side around \( P(A_i) = 0, P(A_j) = 0, P(A_i \cap A_j) = 0 \). The result is:

\[
\frac{z_{ij}}{\delta(0,T)(1 - R)} \approx \frac{T}{\sum_{s=1}^{T} \delta(0,s-1)}(P(A_j) - (1 - S)P(A_i \cap A_j))
\]

It is important to check how good is the approximation for a realistic range of parameters. For several different points in time (every 50 days) between 1/1/2007 and 3/31/2009, I compare the correct spread and the approximated spread, computed using the US yield curve at that time, considering:

- different values of \( P(A_j) \): between 0 and the maximum probability implied by bond data under no liquidity assumptions (\( \max_j \{h_j(0)\} \)).
- different values of \( P(A_i \cap A_j) \): between 0 and \( P(A_j) \)
- different values of \( R \) and \( S \): between 0.1 and 0.4.
The average error (approximated spread-true spread) in the monthly spread is 0.3 basis points, while the median error is 0.2bp. The standard deviation is 0.3bp. This indicates a tendency to overestimate spreads by about 0.5% on average in the simulations, with a standard deviation of 0.4%. The maximum error in absolute value in the parameter range selected was 1% of the CDS spread (about 2bp). Therefore, the approximation is extremely good.

Appendix C - Proofs

C.1 - Proof of Proposition 3

Start from a symmetric LP problem

\[
\begin{align*}
\max & \quad c'p \\
\text{s.t.} & \quad Ap \leq b
\end{align*}
\]

Suppose that \( p^\ast \) is a solution to the problem. Given the definition of symmetry presented in the text, it is clear that \( p^\ast_J \) is also a solution to the problem: \( c'p^\ast = c'p^\ast_J \) and similarly hold for every row of the constraints, for every \( J \).

Now, construct \( p^{**} \) as follows:

\[
p^{**} = \frac{1}{2N} \sum_J p^\ast_J
\]

where the first \( J \) correspond to no permutation, and \( J \) cycles across all permutations of indices \( A_1, ..., A_N \).

Note that it is also possible to construct \( p^{**} \) in the following way, considering the binary representation introduced in Proposition 2. Every \( b_i \) vector has \( O_i \) ones and \( N - O_i \) zeros. Call \( H_i \) the set of all vectors of size \( N \) that have \( O_i \) ones and \( N - O_i \) zeros in different positions. Call \( b_{ih} \) the vector corresponding to element \( h \) from \( H_i \). Then, for every \( i \), construct \( p^{**} \) as:

\[
p^{**}_i = \left( \frac{O_i}{N} \right)^{-1} \sum_{h \in H_i} b_{ih}
\]

From the first construction, it is clear why \( p^{**} \) is a solution to the maximization problem, being just an average of solutions. Plus, \( p^{**} \) is symmetric, which proves the statement of the Proposition.

An example with \( N = 3 \). As explained in Proposition 2, we can construct the probability system \( p^\ast \) as follows:

\[
p^\ast_1 = Pr\{\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3\}
\]

\[
p^\ast_2 = Pr\{\overline{A}_1 \cap \overline{A}_2 \cap A_3\}
\]

\[
p^\ast_3 = Pr\{\overline{A}_1 \cap A_2 \cap \overline{A}_3\}
\]

\[
p^\ast_4 = Pr\{A_1 \cap \overline{A}_2 \cap \overline{A}_3\}
\]

\[
p^\ast_5 = Pr\{A_1 \cap \overline{A}_2 \cap \overline{A}_3\}
\]
\[ p_6^* = \Pr\{A_1 \cap \overline{A}_2 \cap A_3\} \]
\[ p_7^* = \Pr\{A_1 \cap A_2 \cap \overline{A}_3\} \]
\[ p_8^* = \Pr\{A_1 \cap A_2 \cap A_3\} \]

Suppose \( p^* \) solves the maximization problem, and construct \( p^{**} \) as:

\[
\begin{align*}
p_1^{**} &= p_1^* \\
p_2^{**} &= p_3^{**} = p_6^{**} = \frac{p_2^* + p_3^* + p_5^*}{3} \\
p_4^{**} &= p_6^{**} = p_7^{**} = \frac{p_4^* + p_6^* + p_7^*}{3} \\
p_8^{**} &= p_8^*
\end{align*}
\]

\( p^{**} \) solves the maximization problem and is symmetric.

C.2 - Proof of Proposition 4

Given Proposition 3, the following theorem holds, which can be derived as a special case of the bounds presented in Boros and Prekopa (1989), in section 8.

Here, \( S_1 = Nq_1 \) and \( S_2 = \frac{N(N-1)}{2} q_2 \). The bounds are nonlinear functions of \( S_1 \) and \( S_2 \), and take the following form, for \( 3 \leq r \leq N - 1 \).

General lower bound

Call

\[
\begin{align*}
A_1 &= (r-1)N - (r + N - 2)S_1 + 2S_2 \\
A_2 &= -(r-2)S_1 + 2S_2
\end{align*}
\]

- If \( A_2 \leq 0 \)

\[ p_r \geq 0 \]

- If \( A_1 \geq 0 \) and \( A_2 \geq 0 \)

\[ p_r \geq \frac{-(r-2)S_1 + 2S_2}{(N+1-r)N} \]

- If \( A_1 < 0 \)

\[ p_r \geq \frac{(r-1)(r-2i-2) + 2iS_1 - 2S_2}{(i-r+2)(i-r+1)} \]

with

\[ i = \left\lfloor \frac{2S_2 - (r-2)S_1}{S_1 - (r-1)} \right\rfloor \]

where \([x]\) is the largest integer smaller than \( x \).
Note that in general either \( A_1 < 0 \) or \( A_2 < 0 \) in the region that satisfies the necessary consistency relations between \( S_1 \) and \( S_2 \).

**General upper bound** Call

\[
B_1 = rN - (r + N - 1)S_1 + 2S_2 \\
B_2 = -(r - 1)S_1 + 2S_2
\]

- If \( B_1 < 0 \)
  \[ p_r \leq 1 \]
- If \( B_1 \geq 0 \) and \( B_2 \geq 0 \) then
  \[ p_r \leq \frac{(r + N - 1)S_1 - 2S_2}{rN} \]
- If \( B_2 < 0 \)
  \[ p_r \leq \frac{i(i + 1) - 2iS_1 + 2S_2}{(r - i - 1)(r - i)} \]
  where
  \[ i = \left\lfloor \frac{(r - 1)S_1 - 2S_2}{r - S_1} \right\rfloor \]

**Lower bound in the symmetric case**

Call

\[
A_1 = (r - 1)N - (r + N - 2)Nq_1 + N(N - 1)q_2 \\
A_2 = -(r - 2)Nq_1 + N(N - 1)q_2
\]

and remember that \( q_2 \leq q_1 \). Then,

\[ A_2 \leq 0 \iff q_2 \leq \frac{(r - 2)}{(N - 1)}q_1 \]

\[ A_1 \leq 0 \iff q_2 \leq \frac{(r + N - 2)q_1 - r + 1}{N - 1} = \frac{(r - 2)q_1}{N - 1} + \frac{Nq_1 - r + 1}{N - 1} \]

Remember the following relations between \( S_1 \) and \( S_2 \) (or respectively \( q_1 \) and \( q_2 \)):

\[ q_1 \geq q_2 \]

and (Bonferroni Inequality):

\[ S_1 - S_2 \leq 1 \]

so that:

\[ \frac{N(N - 1)}{2}q_2 \geq Nq_1 - 1 \]

or:

\[ \frac{2}{N - 1} \left( q_1 - \frac{1}{N} \right) \leq q_2 \leq q_1 \]
In addition, note that the Bonferroni Inequality is not sharp, so that not all $q_2$ that satisfy that condition are actually compatible with the existence of an underlying probability system. The correct lower bound for $q_2$ given $q_1$ is given by the requirement that the lower bound for $r = 1$ (the probability of the union of all events) is less or equal to one. Call this number $\overline{q_2}$.

The bounds then can be written as follows.

CASE 1: If $q_1 < \frac{r - 1}{N}$

- If $\overline{q_2} \leq q_2 \leq \frac{(r - 2)}{(N - 1)} q_1$
  
  $\frac{p_r}{N} \geq 0$

- If $\frac{(r - 2)}{(N - 1)} q_1 \leq q_2 \leq q_1$
  
  $\frac{p_r}{N} \geq \frac{-(r - 2)Nq_1 + N(N - 1)q_2}{(N + 1 - r)N}$

CASE 2: If $q_1 > \frac{r - 1}{N}$

- If $\overline{q_2} < q_2 \leq \frac{(r - 2)}{(N - 1)} q_1 + \frac{Nq_1 - r + 1}{N - 1}$
  
  $\frac{p_r}{N} \geq \frac{(r - 1)(r - 2i - 2) + 2iNq_1 - N(N - 1)q_2}{(i - r + 2)(i - r + 1)}$

- If $q_2 \geq \frac{(r - 2)}{(N - 1)} q_1 + \frac{Nq_1 - r + 1}{N - 1}$
  
  $\frac{p_r}{N} \geq \frac{-(r - 2)Nq_1 + N(N - 1)q_2}{(N + 1 - r)N}$

Note that in this case, to the left of $\frac{(r - 2)}{(N - 1)} q_1$ we would have $A_2 \leq 0$. However, this number has to be less than $\overline{q_2}$; otherwise, there would be a discontinuity in the solution to the minimization problem, since the bound is decreasing to the right of $\frac{(r - 2)}{(N - 1)} q_1$ and positive, and would be 0 to the left. But we know that the bound is a continuous function of the constraints $q_1$ and $q_2$.

Upper bound in the symmetric case

Call

\[ B_1 = rN - (r + N - 1)Nq_1 + N(N - 1)q_2 \]

\[ B_2 = -(r - 1)Nq_1 + N(N - 2)q_2 \]

The key points are whether

\[ \frac{(r + N - 1)}{N - 1} q_1 - \frac{r}{N - 1} > \frac{(r - 1)}{(N - 2)} q_1 \]

or

\[ \frac{(N - 2)(r + N - 1) - (r - 1)(N - 1)}{(N - 2)} q_1 > r \]
\[ q_1 > \frac{N - 2}{N(N - 2) - r + 1} r \]

Note that in this case we have the requirement that \( q_2 \) is not so low that the probability of the intersection given the bounds comes out less than 0. I.e. we need that \( q_2 \) is at least \( \hat{q}_2 \) where \( \hat{q}_2 \) is such that the upper bound with \( r = N \) is greater than 0.

Then we have:

CASE 1: If \( q_1 > \frac{N - 2}{N(N - 2) - r + 1} r \)

- If \( \hat{q}_2 \leq q_2 < \frac{(r+N-1)}{N-1} q_1 - \frac{r}{N-1} \)
  \[ p_r \leq 1 \]
- If \( q_2 \geq \frac{(r+N-1)}{N-1} q_1 - \frac{r}{N-1} \)
  \[ p_r \leq \frac{(r + N - 1)Nq_1 - N(N-1)q_2}{rN} \]

CASE 2: If \( q_1 < \frac{N - 2}{N(N - 2) - r + 1} r \)

- If \( \hat{q}_2 \leq q_2 < \frac{(r-1)}{(N-2)} q_1 \)
  \[ p_r \leq \frac{i(i + 1) - 2iNq_1 + N(N - 1)q_2}{(r - i - 1)(r - i)} \]
  where
  \[ i = \left[ \frac{(r - 1)Nq_1 - 2N(N-1)q_2}{r - Nq_1} \right] \]
- If \( q_2 \geq \frac{(r-1)}{(N-2)} q_1 \)
  \[ p_r \leq \frac{(r + N - 1)Nq_1 - N(N-1)q_2}{rN} \]

Appendix D - Details on the Robustness Tests

D.1 - Assumptions on S

As evident from equation (2), the recovery rate of a CDS in case of double default, \( S \), affects positively the value of that security. In addition, a higher recovery rate \( S \) implies a higher sensitivity of the CDS spreads to changes in the joint default risk. A recovery rate of zero \( (S = 0) \) means that counterparty risk has the highest impact on CDS spreads, while a recovery rate of 1 means that joint default risk has no effect on the spread.

To understand the effect of assumptions on \( S \) on the measure of systemic risk presented in this paper, remember that the upper bound on systemic risk is attained by the most correlated probability system that satisfies the constraints:

\[ P(A_i) \leq h_i(\gamma_i^p) \]
\[ P(A_i) - (1 - S) \left( \frac{1}{N-1} \sum_{j \neq i} P(A_i \cap A_j) \right) = b_i \]

where \( b_i = z_i \left[ \sum_{T=1}^{T_i} \frac{\delta(0,s-1)}{T_0(s,T)(1-R)} \right] \).

Intuitively, for given \( S \), one would obtain the most correlated probability system by setting \( P(A_i) \) as high as possible (up to the constraint \( h_i(\gamma_i t) \)) for all banks, and then increasing the term \( \frac{1}{N-1} \sum_{j \neq i} P(A_i \cap A_j) \) to match the CDS spreads \( (b_i) \). Counterparty risk would explain the whole bond/CDS basis, and a higher recovery \( S \) implies that a higher joint default probability is needed to match it, increasing the upper bound on systemic risk. This intuitive reasoning, however, does not take into account the internal restrictions of consistency of the probability system. These can be best understood in the context of a symmetric network.

For a symmetric network, using the notation of Section 3, call the marginal probabilities of default \( q_1 \) and the pairwise joint probabilities of default \( q_2 \). The previous constraints become:

\[ q_1 \leq h \]
\[ q_1 - (1 - S)q_2 = b \]

where \( h \) is the (common) upper bound on the marginal probability of default and \( b \) is the (common) \( b_i \).

To maximize systemic risk, we would intuitively set \( q_1 = h \), and then \( q_2 \) will be set to match CDS spreads:

\[ q_2 = \frac{q_1 - b}{1 - S} \quad (10) \]

For given \( q_1 \), \( q_2 \) is increasing in \( S \), and systemic risk with it. This captures the intuition that a higher recovery rate of CDSs implies that higher counterparty risk is needed to explain the same bond/CDS basis.

In fact, this effect is at play only when \( S \) is small enough. As \( S \) grows, \( q_2 \) keeps increasing, and at some point it will reach the level \( q_2 = q_1 \). At that point, the internal consistency of the probability system kicks in, preventing further increases: it would violate the implicit constraint \( q_2 \leq q_1 \).

What happens then if \( S \) increases further? The only way to satisfy the constraints is to lower \( q_1 \) below \( h \); for \( q_1 = h \) there might exist no probability system able to satisfy both constraints: matching the CDS spread and satisfying internal consistency. Instead, with a lower \( q_1 \), it is possible to set \( q_2 \) to be equal to \( q_1 \) and satisfy the CDS constraint, so that:

\[ q_2 = q_1 = \frac{b}{S} \]

which is decreasing in \( S \). This means that for these values of \( S \), the bond/CDS basis is too large to be explained by counterparty risk. Even at the upper bound on systemic risk, liquidity has to explain part of the basis. In a symmetric system, then, the bounds on systemic risk first increase and then decrease with \( S \).

\[ I \text{ focus on the upper bound for the probability of at least } r > 1 \text{ events occurring. Following the analysis reported} \]
\[ \text{in section 3, the same argument holds for the lower bound for the probability that at least 1 institution defaults, since that is achieved for a very correlated system. It is easy to see why the results for the lower bound for } r > 1 \text{ and the upper bound for } r = 1 \text{ do not depend on } S: \text{ these bounds look for the least correlated system, which can always be obtained by setting the marginal default probabilities at the levels implied by the CDS spreads and attributing the bond/CDS basis entirely to liquidity. Also, in an asymmetric network the solutions to the problems for different } r \text{ will not be the same, even though they all present a high degree of pairwise correlations.} \]

61
These forces play out in similar but nonlinear ways for asymmetric networks. In that case, the asymmetry in the bond/CDS basis across banks means that the upper bounds on marginal probabilities (that are obtained from bond prices) will bind for some banks and not for others. The overall effect on the bounds is difficult to describe analytically but can be tested numerically.

D.2 - Stochastic recovery rate

This section shows that if recovery rates are stochastic, the change in bond and CDS prices can be decomposed as follows. Starting from the prices obtained in case of constant recovery rate $R$ (Section 4), the prices are adjusted changing the recovery rate to a lower value $R_L$. In addition, the price of the bond and the present value of CDS payments are shifted in opposite directions by an amount $Y$, which is the same to a first-order approximation. Both these changes have a minor impact on the bond/CDS basis and therefore should not change the main results of the paper.

To see this, assume that the recovery rate on bonds is $R_H$ is one bank defaults alone, but is $R_L$ if more than one bank defaults at the same time. Call $X_i = \cup_{k \neq i} A_k$ the event of at least one default among the banks different from $i$, and similarly $X_{ij} = \cup_{k \neq i,j} A_k$. Then, it is easy to see that the bond price becomes (set liquidity to 0 for simplicity)

$$B(0, T) = c \left( \sum_{t=1}^{T} \delta(0, t)(1 - P(A_i))^t \right) + \delta(0, T)(1 - P(A_i))^T + R_H \left( \sum_{t=1}^{T} \delta(0, T)(1 - P(A_i))^{t-1}P(A_i \cap X_i) \right) + R_L \left( \sum_{t=1}^{T} \delta(0, T)(1 - P(A_i))^{t-1}P(A_i \cap X_i) \right)$$

We can rewrite this expression as:

$$B(0, T) = c \left( \sum_{t=1}^{T} \delta(0, t)(1 - P(A_i))^t \right) + \delta(0, T)(1 - P(A_i))^T + R_L \left( \sum_{t=1}^{T} \delta(0, T)(1 - P(A_i))^{t-1}P(A_i) \right) + Y_{bond}$$

where

$$Y_{bond} = \sum_{t=1}^{T} \delta(0, T)(1 - P(A_i))^{t-1}P(A_i \cap X)(R_H - R_L)$$

The price of the bond is now equal to the price of the bond in case that the recovery rate is constant and equal to $R_L$ plus the last term, which is the present value of the additional recoveries in case $i$ defaults alone.

Similarly, the CDS spread becomes:

$$\sum_{s=1}^{T} \delta(0, s-1)(1 - P(A_i \cup A_j))^{s-1}z_{ji} =$$

$$= \sum_{s=1}^{T} (1 - P(A_i \cup A_j))^{s-1}\delta(0, T)[P(A_i \cap \overline{A}_j \cap \overline{X}_{ij})(1 - R_H)$$
The right-hand side can be rewritten as follows (noting that \( P(A_i \cap A_j \cap X) = P(A_i \cap X_i) \)):

\[
\left\{ \sum_{s=1}^{T} (1 - P(A_i \cup A_j))^{s-1} \delta(0, T) (P(A_i) - (1 - S)P(A_i \cap A_j)) (1 - R_L) \right\} - Y_{cds}
\]

where

\[
Y_{cds} = \sum_{s=1}^{T} (1 - P(A_i \cup A_j))^{s-1} \delta(0, T) P(A_i \cap X_i) (R_H - R_L)
\]

Just as in the case of bond prices, the present value of payments from the contract is equal to the one that obtains if the recovery rate is constant at \( R_L \), minus an amount \( Y_{cds} \) that is approximately the same as \( Y_{bond} \). Therefore, the effects coming from \( Y_{bond} \) and \( Y_{cds} \) shift the yield of the bond and the CDS spread down by similar amounts. In fact, because \((1 - P(A_i \cup A_j)) \leq (1 - P(A_i))\), \( Y_{cds} \leq Y_{bond} \), and therefore the effect on the yield might in fact be larger than the effect on the CDS spread, leading to a net reduction of the basis and therefore a strengthening of the results on systemic risk.

D.3 - Alternative pricing model

Assume that at every time \( t \) agents assume a deterministic hazard rate of the form:

\[
h_{t+s} = (1 - \rho_t)\bar{h}_t + \rho_t h_{t+s-1}
\]

with a certain \( \bar{h}_t, \rho_t, \) and \( h_t \) determined at time \( t \).

Note that

\[
h_{t+s} = (1 - \rho_t)(1 + \ldots + \rho_t^{s-2})\bar{h}_t + \rho_t^{s-1}h_t
\]

for \( s \geq t \).

The probability of surviving until \( t + r \) is

\[
H_t(t + r; h_t, \rho_t) = (1 - h_t)\ldots(1 - h_{t+r})
\]

From the cross section of outstanding bonds, we can then estimate at each \( t \) the three parameters \( h_t, \rho_t, \bar{h}_t \).

Since CDS spreads depend on the process of joint hazard rate of default, but we only observe the price of the 5-year CDS, I assume that the shape of the joint default hazard rate \( h_{ij} \) is similar to that of the marginal hazard rate of the reference entity \( i \) (similar results hold if we assume that it inherits the shape estimated from the bond prices of the seller, or a combination of both). In particular, after having estimated the three parameters for the hazard rate of bank \( i \), I define

\[
\alpha^i_t = \frac{h^i_t}{\bar{h}^i_t}
\]
so that:

\[ h_{t+s} = (1 - \rho_t^{t+s-1}T_t + \rho_t^{t+s-1}h_t = (1 - \rho_t^{t+s-1})\alpha_t h_t + \rho_t^{t+s-1}h_t = (\alpha_t - \rho_t^{t+s-1})\alpha_t + \rho_t^{t+s-1}h_t \]

Call \( h^i_t \) the probability that \( i \) defaults, similarly for \( j \) and finally \( h^{ij}_{t} \) the probability of joint default. Assume that:

\begin{align*}
    h^i_{t+s} &= (\alpha^i_t - \rho^{i,t+s-1}_t\alpha^i_t + \rho^{i,t+s-1}_t)h^i_t \\
    h^j_{t+s} &= (\alpha^j_t - \rho^{j,t+s-1}_t\alpha^j_t + \rho^{j,t+s-1}_t)h^j_t \\
    h^{ij}_{t+s} &= (\alpha^{ij}_{t} - \rho^{ij,t+s-1}_t\alpha^{ij}_{t} + \rho^{ij,t+s-1}_t)\]

And assume \( \alpha^{ij}_{t} = \alpha^i_t \) and \( \rho^{ij}_{t} = \rho^j_t \). Note that this reduces the CDS spread to a function of only one parameter: \( h^{ij}_{t} \). Note also that this requires \( h^i_t > 0 \) for \( \alpha^i_t \) to be defined. Therefore, I impose a lower bound on \( h^i_t \) of \( 10^{-7} \).

Define

\[ H^{ij}_{t}(t+s; h_t, \rho_t) = (1 - h^i_t)(1 - h^{ij}_{t+1})...(1 - h^{ij}_{t+s}) \]

which is the probability of having no credit events until time \( t+s \), and

\[ H^{ij}_{t}(t; h_t, \rho_t) = 1 \]

The CDS spread at time \( t \) is:

\[ z^{ij}_{t} = \delta(t,T)(1 - R) \left[ \sum_{s=1}^{T} H^{ij}_{t}(t+s-1; h_t, \rho_t)(h^{ij}_{t+s} - (1 - S)h^{ij}_{t+s}) \right] \]

A linear approximation around \( h^{ij}_{t} = 0 \) yields:

\[ \frac{(N - 1)}{G_i} \frac{z^{ij}_{t}}{\delta(t,T)(1 - R)} \approx (N - 1)h^{i}_{t} + \sum_{j \neq i} \frac{G^{ij}_{t}}{G_i}h^{ij}_{t} \]

with

\[ G_i = \left[ \alpha^i_t(T-t) + (1 - \alpha^i_t)\frac{1 - \rho^{i}_{t}(T-t)}{1 - \rho^{i}_{t}} \right] \]

\[ G_{ij} = -(1 - S) \left[ \alpha^{ij}_{t}(T-t) + (1 - \alpha^{ij}_{t})\frac{1 - \rho^{ij}_{t}(T-t)}{1 - \rho^{ij}_{t}} \right] \]

which is again linear in the marginal and pairwise default probabilities and can be used in the LP formulation.
D.4 Currency assumptions

To simplify the discussion, consider one-period bonds and CDSs denominated in different currencies. Call $m_{se}$ the stochastic discount factor of a US investor in state $(s, e)$ where $s$ indicates whether default of firm $i$ ($s = i$ or 0) occurs or not and $e$ is the exchange rate. Call $f(s, e)$ the joint density function $s$ and $e$. To simplify notation, rewrite $f(s, e)$ as:

$$f(s, e) = \pi_s f_s(e)$$

Note that

$$\pi_0 E[m_{se} | s = 0] + \pi_i E[m_{se} | s = i] = E[m_{se}]$$

For a dollar-denominated risky bond (R is the recovery rate), the dollar price is:

$$p_i^\$ = \pi_0 E[m_{se} | s = 0] + R\pi_i E[m_{se} | s = i]$$

$$= E[m_{se}] - (1 - R)\pi_i E[m_{se} | s = i]$$

Now consider a euro-denominated bond issued by the same firm, and of equal seniority. Arbitrage requires (calling $e_0$ the time-0 exchange rate):

$$p_i^E e_0 = \pi_0 E[e \cdot m_{0e} | s = 0] + R\pi_i E[e \cdot m_{ie} | s = i]$$

$$= E[e \cdot m_{se}] - (1 - R)\pi_i E[e \cdot m_{se} | s = i]$$

The prices of the respective risk-free securities are:

$$t^$ = E[m_{se}]$$

$$t^E e_0 = E[e \cdot m_{se}]$$

Combining defaultable and risk-free bonds we get:

$$p_i^\$ = t^$ \left( 1 - (1 - R)\pi_i \frac{E[m_{se} | s = i]}{E[m_i]} \right)$$

$$p_i^E e_0 = t^E e_0 \left( 1 - (1 - R)\pi_i \frac{E[e \cdot m_{se} | s = i]}{E[e \cdot m_{se}]} \right)$$

We can then use either bond to estimate the risk-neutral probability of default of firm $i$

$$P(A_i) = \pi_i \frac{E[m_{se} | s = i]}{E[m_{se}]}$$

discounting cash flows by the appropriate risk-free rates as long, as the following condition holds:

$$\frac{E[e \cdot m_{se} | s = i]}{E[m_{se} | s = i]} = \frac{E[e \cdot m_{se}]}{E[m_{se}]}$$

which requires that the relative price of domestic and foreign risk-free securities is the same as the relative price of
domestic and foreign state-contingent securities that pay off if \( i \) defaults.

Now, consider the case of a CDS written by one bank on the default of another bank. The CDS is written on a European bank \((i)\) but the counterparty \((j)\) is American. The contract is denominated in euros. In this case, \( s \) represents all the combinations of default of the two banks, and can be \( i \) (only \( i \) defaults), \( j, ij \) (both default), and \( 0 \) (none defaults).

The CDS contract costs \( z_{ji} \) euros. So we must have

\[
z_{ji}e_0 = (1 - R)\pi_i E[e \cdot m_{se}|s = i] + (1 - R)S\pi_{ij} E[e \cdot m_{se}|s = ij]
\]

or:

\[
z_{ji}e_0 = E[e \cdot m_{se}] \left((1 - R)\pi_i \frac{E[e \cdot m_{se}|s = i]}{E[e \cdot m_{se}]} + (1 - R)S\pi_{ij} \frac{E[e \cdot m_{se}|s = ij]}{E[e \cdot m_{se}]}ight)
\]

Again then, as long as the European yield curve to discount cash flows for Euro-denominated CDSs the condition sufficient is similar to before:

\[
\frac{E[e \cdot m_{se}|s]}{E[m_{se}|s]} = \frac{E[e \cdot m_{se}]}{E[m_{se}]}
\]

for every default event in \( s \).
Tables

Table 1

<table>
<thead>
<tr>
<th>Institution</th>
<th>Avg valid bonds</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
<th>2010</th>
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<tr>
<td>Abn Amro</td>
<td>3.3</td>
<td>1.8</td>
<td>2.2</td>
<td>4.0</td>
<td>4.4</td>
<td>3.0</td>
<td>3.5</td>
<td>5.3</td>
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<td>3.0</td>
<td>2.4</td>
<td>2.5</td>
<td>9.1</td>
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<td>3.0</td>
<td>3.9</td>
<td>6.7</td>
<td>18.0</td>
<td>22.6</td>
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<td>25.4</td>
<td>29.3</td>
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<td>35.3</td>
<td>41.8</td>
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<td>11.6</td>
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<td>40.4</td>
<td>49.0</td>
<td>57.0</td>
<td>63.0</td>
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<td>11.1</td>
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<td>26.5</td>
<td>31.4</td>
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<td>-</td>
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<tr>
<td>Merrill Lynch</td>
<td>35.7</td>
<td>22.7</td>
<td>33.0</td>
<td>38.4</td>
<td>43.0</td>
<td>44.0</td>
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<td>-</td>
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<td>Morgan Stanley</td>
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Note: first column reports average number of bonds for each institution that are used for the estimation of marginal default probabilities. Columns 2-8 break this number down by year.

Table 2

<table>
<thead>
<tr>
<th>Institution</th>
<th>Avg CDS spread</th>
<th>Std CDS spread</th>
<th>Min spread</th>
<th>Max spread</th>
<th>Avg basis</th>
<th>Std basis</th>
<th>Min basis</th>
<th>Max basis</th>
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<tbody>
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<td>0.0005</td>
<td>0.0191</td>
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<td>0.0032</td>
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Note: table reports descriptive characteristics on the CDS spread and the yield spread for the 15 institutions in the sample. The yield spread is computed as the linearly interpolated yield for a 5-year maturity bond in excess of the corresponding Treasury rate.
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Figures

Figure 1: Average bond yield and CDS spread
Figure 2: Example of the relation between low-order and high-order probabilities
Figure 3: Construction of the Linear Programming representation
Figure 4: Bond yield spreads and CDS spreads
Figure 5: Bounds on different systemic events
Figure 6: Bounds with liquidity assumptions
Figure 7: Bounds with average information
Figure 8: Bounds with average information and liquidity assumptions
Figure 9: Part of the network in the high systemic risk scenario as of 08/4/2008
Figure 10: Marginal and pairwise default probabilities
Figure 11: Individual contributions to systemic risk
Figure 12: Bounds with mean-reverting hazard rate process
Figure 13: Bounds computed using only American firms