A Logic for Two-Dimensional Semantics

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Abstract

Two-dimensional semantics is a theory in the philosophy of language that provides an account of meaning which is sensitive to the distinction between necessity and apriority. While this theory is usually presented in an informal manner, I take some steps in formalizing it in this paper. To do so, I define a semantics for a propositional modal logic with operators for the modalities of necessity, actuality, and apriority that captures the relevant ideas of two-dimensional semantics. I use this to answer one of the criticisms of two-dimensional semantics, namely that its use of two intensions is problematic for specifying a compositional semantics, and apply it to the so-called nesting problem. Finally, I present a sound and complete proof system for the logic.

1 Two-Dimensional Semantics

The term “two-dimensional semantics” has been applied to a variety of theories, an overview of which can be found in Schroeter (2010). In this paper, I will only be concerned with the variant proposed by David Chalmers, which he calls epistemic two-dimensional semantics, e.g., in Chalmers (2004). It can be seen as a framework dealing with meaning and its connection to necessity and apriority, as well as to issues of reference determination and context dependence. Chalmers has also used it in an argument concerning the problem of consciousness, in Chalmers (1996), and in a discussion of propositional attitude ascriptions, in Chalmers (2011).

Central to two-dimensional semantics is the connection between meaning and the modalities of necessity and apriority. Concerning these two notions, Chalmers simply notes that possible (not necessarily not the case) is what might have been the case Chalmers (2004, p. 210), and explains that roughly, a priori is what “can be conclusively non-experientially justified on ideal rational reflection” Chalmers (2004, p. 208). In standard intensional semantics, the meaning of an expression is modeled by a function, called its intension, that maps possible worlds to extensions. Taking the extensions of sentences to be truth values, necessity is then analyzed as having an intension that maps every world to the truth value true. One of the central claims of two-dimensional semantics is that both necessity and apriority can be analyzed in such a manner. To do so, two-dimensional semantics postulates two kinds of possibilities. The first are the familiar (possible) worlds, which can be understood as ways the world could be. The second are called scenarios, and can be understood as ways the world might be, given what can be known a priori. It is also assumed that every
scenario is associated with a unique corresponding world, where two scenarios might be associated with the same world. The evaluation of expressions is now relativized to both worlds and scenarios, and it is this kind of double-indexing from which the theory takes its name. More precisely, two-dimensional semantics models the meaning of an expression by a function that maps every pair containing a world and a scenario to an extension. I will call this function the two-dimensional intension.

From such a two-dimensional intension, two (one-dimensional) intensions are derived: The primary intension is defined as the function that maps every scenario to what the pair containing it and its possible world is mapped to by the two-dimensional function. The secondary intension is defined as the function that maps every world to what the pair containing the actualized scenario (the scenario that is in fact realized) and that world is mapped to by the two-dimensional function. Apriority is then analyzed as having a constant primary intension mapping every scenario to true, and necessity is analyzed as having a constant secondary intension mapping every world to true. We can visualize two-dimensional intensions as tables or matrices, where worlds are listed on the horizontal axis, scenarios are listed on the vertical axis in corresponding order, and the appropriate extensions are written in the cells. This is why two-dimensional intensions are also called two-dimensional matrices. We can then visualize the analysis of necessity as requiring the table to contain true throughout the horizontal at the actualized scenario. Similarly, we can visualize the analysis of apriority as requiring the table to contain true throughout the diagonal.

What I have said so far is of course only an outline of two-dimensional semantics. E.g., more needs to be said about worlds and scenarios, especially as the account of primary intensions assumes that every scenario is associated with its world. Another important aspect about which I haven’t said anything so far is reference determination, which of course also plays a crucial role. For this paper, these details will not be important, so rather than filling in more of them, I will give an example. Let me use “Hesperus is Phosphorus”, assuming that Kripke (1980 [1972]) is correct in claiming that this is necessary and a posteriori. Let s and t be scenarios, and w and v their associated worlds, s being the actualized scenario and w the actual world. Then ignoring the other scenarios and worlds, the two-dimensional intension of “Hesperus is Phosphorus” might look like this:

<table>
<thead>
<tr>
<th></th>
<th>w</th>
<th>v</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>true</td>
<td>true</td>
</tr>
<tr>
<td>t</td>
<td>false</td>
<td>false</td>
</tr>
</tbody>
</table>

Since s is the actualized scenario, the secondary intension of “Hesperus is Phosphorus” is the constant function to true, which represents the fact that it is necessary. But the primary intension contains a false, which represents the fact that it is not a priori.

Two-dimensional semantics has been criticized on a number of grounds. One of them is that it “generates difficulties for a systematic combinatorial semantics” Schroeter (2010, section 2.4.2). (Schroeter only presents this criticism; see the passage cited for references to authors who endorse it.) In particular, this line of argument has been pursued by Scott Soames, who has given a series of arguments which are supposed to exhibit problems of this kind for two-
dimensional semantics. One of these arguments, described in Soames (2005, see *Argument 5* on pp. 278–279), was discussed in a more abstract form in Dever (2007, see *Version 2* on p. 11), and has been reformulated in its most perspicuous form in Chalmers (2011, endnote 25). In Chalmers’s rendition, it is the following schematic argument, consisting of two premises, (A1) and (A2), and a conclusion, (A3):

(A1) If it is a priori that $p$, then it is necessarily a priori that $p$.

(A2) Necessarily, if it is a priori that $p$, then $p$.

(A3) If it is a priori that $p$, then it is necessary that $p$.

The conclusion follows from the premises by standard modal reasoning: As (A2) is a necessitated implication, we can infer that the necessitated antecedent implies the necessitated consequent, from which (A3) follows with (A1) by chaining of implications. According to Soames’s discussion of his version, two-dimensional semantics entails the truth of the first premise. As the claim that the conclusion has false instances is part of two-dimensional semantics, and Soames assumes the truth of the second premise, he takes the argument to be a refutation of two-dimensional semantics Soames (2005, p. 325). Dever does not draw such a strong conclusion, but notes that the problem raises the “important and imposing challenge” of giving “a detailed story about how [primary and secondary intensions] compose and interact” Dever (2007, p. 16). While I agree with Dever in this, it should be noted that (A1) seems quite plausible independently of two-dimensional semantics, so the argument poses a problem for everyone who accepts the existence of a priori contingencies. Since the crucial element of the argument is the nesting of “a priori” inside the scope of “necessarily”, I call this the nesting problem.

### 2 A Formalization

The alleged difficulties in providing a systematic semantics as well as the more specific nesting problem motivate spelling out the details of two-dimensional semantics in a rigorous form. By giving a formal system that captures the central claims of two-dimensional semantics, we can show that the aspects that are being modeled can be systematized. If this system provides an explicit treatment of the modalities of necessity and apriority as well as the usual Boolean connectives, we can also represent the nesting problem in it, which might shed some light on how to solve it. Finally, it is interesting just to see how the modalities of necessity and apriority relate and interact according to two-dimensional semantics. This might also provide a way of testing two-dimensional semantics, by checking whether these relations and interactions are in accord with our understanding of necessity and apriority. The most natural way of developing such a formal system is in the form of a modal logic.

One might wonder whether such a logic has not already been defined. And in fact, for some of the uses just mentioned, this is the case. E.g., the logics in both Segerberg (1973) and Davies and Humberstone (1980) show that relativizing truth to two different indices does not prevent us from giving a formal semantics. But neither of them provides a formalization of two-dimensional semantics, or is
even intended to represent the notion of apriority, which is of crucial importance here (see also Humberstone (2000, p. 278)). Similarly, Michels (2011) defines a formal system to analyze another one of Soames’s arguments against two-dimensional semantics. While this can be used to argue that some aspects of two-dimensional semantics can be systematized, his system does not include an operator for apriority either, and can therefore not be used to represent the nesting problem, nor to systematically investigate the relations between necessity and apriority.

The one text that presents a logic that fits the description given above is Restall (2011). Essentially, Restall presents the same logic as the one defined below, since apart from using different symbols, he uses the same syntax, and his semantics is easily seen to be equivalent to the one considered here. But the present paper differs from Restall’s in both philosophical and technical respects. The main philosophical difference is that here, the model theory is based on the ideas of two-dimensional semantics, and used to define a logic that is then applied to the nesting problem, whereas Restall starts with independently motivated rules of a kind of sequent calculus, which he then uses to define a model theory that is intended to motivate some aspects of two-dimensional semantics. The main technical difference concerns the proof system, as I use a more traditional axiomatization than Restall.

So let me now start to develop a formal logic with modal operators for necessity and apriority according to the ideas of two-dimensional semantics. Since quantification in modal contexts is a notoriously thorny issue, and one that is not essential for present purposes, I will consider a propositional logic here. As the relation of indexicals to necessity and apriority in two-dimensional semantics is also interesting, I am going to include a modal operator representing “actually” in the logic. Accordingly, I will consider a propositional modal language with operators for necessity, apriority and actuality. I use lower-case letters like $p$ and $q$ as proposition letters; $\neg$ and $\land$ as primitive Boolean operators; and $\Box$ for necessity, $A$ for apriority and $@$ for actuality. I use $\diamond$ for the dual of $\Box$ ($\neg\Box\neg$), which is read possibly, and $C$ for the dual of $A$ ($\neg A\neg$), which I read conceivably (but you may just read as not a priori that not, if you do not think that conceivability stands in this relation to apriority).

To try to capture the ideas of two-dimensional semantics in a formal semantics, I will start with a class of Kripke frames. A Kripke frame for our language is a tuple $\langle W, R_{\Box}, R_{@}, R_{A} \rangle$, where $W$ is a set and any $R_{\Delta}$ is a binary relation on $W$. (I use $\Delta$ as a variable ranging over $\Box$, $@$ and $A$.) A Kripke model based on such a frame is obtained by adding a valuation function $V$, which is a function that maps every proposition letter to a subset of $W$. We call $W$ the set of points, and the relations $R_{\Delta}$ the accessibility relations. Note that points need not represent possible worlds, and in fact will not do so in the class of frames used below. Truth of a formula $\varphi$ in a model $M = \langle W, R_{\Box}, R_{@}, R_{A}, V \rangle$ is relativized to a point $w$, and written $M, w \models \varphi$. This is defined recursively with the following clauses:

$M, w \models p$ iff $w \in V(p)$

$M, w \models \neg \varphi$ iff not $M, w \models \varphi$

$M, w \models \varphi \land \psi$ iff $M, w \models \varphi$ and $M, w \models \psi$

$M, w \models \Delta \varphi$ iff $M, v \models \varphi$ for all $v \in W$ such that $wR_{\Delta}v$
Modal logics can be characterized by classes of frames. For such a class $C$, we define a formula $\phi$ to be valid on $C$, written $C \models \phi$, if it is true at any point in any model based on any frame in $C$; and $\phi$ to be a consequence of a set of formulas $\Gamma$ over $C$, written $\Gamma \models_C \phi$, if it is true at any point in any model based on any frame in $C$ at which all elements of $\Gamma$ are true.

I will now define a class of frames based on the ideas of two-dimensional semantics and call them matrix frames. As two-dimensional intensions assign truth values to sentential expressions relative to world-scenario pairs, it would be natural to use frames in which the set of points is the Cartesian product of two sets, one representing worlds and one representing scenarios. However, for simplicity, I will use the same set to represent both worlds and scenarios, and therefore use frames in which the set of points is the Cartesian square of some set. (The Cartesian square of a set $S$ is the set of pairs $(x, y)$ with $x, y \in S$.) This is a simplification, since it is plausible that there are several scenarios that are associated with the same world. I will show in Corollary 15 below that this simplification makes no difference to the logic of the language considered here.

Thinking of the Cartesian square of a set as a table analogously to how two-dimensional intensions were illustrated above, we want to interpret necessity as truth throughout the horizontal and apriority as truth throughout the diagonal. If we assume an account of indexicals along the lines of Kaplan (1989) and take scenarios to play the role of contexts of utterance as well, then we want to interpret actuality as truth at the world of the scenario, i.e., at the intersection of the horizontal and the diagonal. As the evaluation clause of modal operators in Kripke semantics universally quantifies over the points accessible from the point of evaluation, we can capture these ideas by defining the accessibility relations as illustrated in the following picture of a frame for the upper middle element:

![Diagram of a frame]

More formally, we give the following definition, making the arbitrary choice of taking the first element of a pair to represent the world, and the second the scenario:

**Definition 1.** A matrix frame is a frame $\mathfrak{F} = \langle W, R_\Box, R_\@, R_A \rangle$, where $W = S \times S$ for some set $S$, and the relations are given by the following conditions:

- $(x, y) R_\Box (x', y')$ iff $y = y'$
- $(x, y) R_\@ (x', y')$ iff $y = y'$ and $x' = y'$
- $(x, y) R_A (x', y')$ iff $x' = y'$

We say that $\mathfrak{F}$ is based on $S$. Let $M$ be the class of matrix frames.
There is a blatant problem for using $M$ as a semantics: As apparent from the illustration above, in all but the simplest matrix frames, the accessibility relation $R_A$ used to interpret apriority is not reflexive, which means that the formula $Ap \rightarrow p$ is not valid on $M$. But clearly, if $p$ is a priori then $p$ is the case, and this should be captured by a good logic of apriority. However, $R_A$ is reflexive on points on the diagonal, i.e., points $\langle x, x \rangle$ for any $x$, which suggests that we can fix this problem by only taking these points into consideration in the definition of validity and consequence. This might seem philosophically ad hoc and technically inconvenient. But philosophically, there is good reason only to consider these points, for points outside the diagonal represent world-scenario pairs in which the world isn’t the one associated with the scenario. In a sense, these are points of evaluation one can’t find oneself in. The restriction is further motivated by the fact that it mirrors a restriction that is standard in logics of indexicals. Using the terminology introduced in Crossley and Humberstone (1977), it is analogous to adopting real-world validity, which is contrasted with general validity. In restricting validity to the diagonal, we thus also follow Kaplan (1989), and ensure that the logic treats necessity and actuality according to real-world validity, which counts formulas like $\Box p \rightarrow p$ as valid. Although the restriction makes the logic somewhat more difficult to handle, it is technically not too unusual. In fact, it perfectly fits a variant of Kripke frames which are called frames with distinguished elements, in short FWDE, in Segerberg (1971). They are structures like Kripke frames, except that a subset of the set of points is added as a further component. The points in this set are called the distinguished points or elements. Models are obtained as in the case of frames by adding a valuation function, and the definition of the truth of a formula at a point stays the same as well. Only the definitions of validity and consequence are changed by restricting them to distinguished points. That is, for any class of FWDEs $C$, a formula $\varphi$ is valid on $C$ if it is true at any distinguished point in any model based on any FWDE in $C$; and $\varphi$ is a consequence of a set of formulas $\Gamma$ over $C$ if it is true at any distinguished point in any model based on any FWDE in $C$ at which all elements of $\Gamma$ are true. To distinguish frames and their models from FWDEs and their models, I use two different fonts, writing $\mathfrak{F}$ and $\mathfrak{M}$ for the former and $\mathcal{F}$ and $\mathcal{M}$ for the latter. Logics characterized by classes of Kripke frames are normal, which means that they contain classical propositional logic and the distributivity axioms $K_\Delta = \Delta(p \rightarrow q) \rightarrow (\Delta p \rightarrow \Delta q)$, and are closed under modus ponens, uniform substitution and generalization (the rule saying that if $\varphi$ is a theorem, so is any $\Delta \varphi$). Not all logics characterized by classes of FWDEs are normal, since they may not be closed under the rule of generalization. But they are quasi-normal, which means that they contain the smallest normal modal logic $K$ and are closed under the rules of modus ponens and uniform substitution.

We can now give a more adequate formal semantics with the class of FWDEs obtained from adding the points on the diagonal as distinguished points to each matrix frame. I call these matrix FWDEs:

**Definition 2.** A matrix FWDE is an FWDE $\mathcal{F} = \langle W, R_\Box, R_\Box, R_A, D \rangle$, such that $\langle W, R_\Box, R_\Box, R_A \rangle$ is a matrix frame based on a set $S$ and $D = \{ \langle x, x \rangle : x \in S \}$. Let $\mathcal{M}D$ be the class of matrix FWDEs.

I will use $\mathcal{M}D$ as the formal semantics of our propositional modal logic which is supposed to capture two-dimensional semantics. Before discussing some of
the more general aspects of this logic, let me give an example and discuss the
validity of some philosophically interesting formulas. Consider again “Hesperus
is Phosphorus” and the two scenarios and worlds used above. To give a formal
representation of its two-dimensional intension, we take a matrix $\mathfrak{M}$ based on
a two-element set, say $\{0, 1\}$. Note that the set of points $W$, the relations $R_\bigcirc$, $R_\bigcirc$, $R_A$ and the set of distinguished points $D$ are determined by the choice of
this set. E.g., $D = \{(0, 0), (1, 1)\}$. We can turn this into a model $\mathcal{M}$ by adding
a valuation function $V$, intending $p$ to stand for “Hesperus is Phosphorus”, we let $V(p) = \{(0, 0), (1, 0)\}$. We can then verify that $\mathcal{M}, (0, 0) \vDash \bigcirc p$ and
$\mathcal{M}, (0, 0) \vDash \neg A p$.

It is also interesting to check whether some philosophically relevant principles
are valid. Given that it is part of two-dimensional semantics that apriority does
not imply necessity and necessity does not imply apriority, it is not surprising
that neither $A p \rightarrow \bigcirc p$ nor $\bigcirc p \rightarrow A p$ is valid on $\mathcal{M}$. While necessity does
not imply apriority, one might think that $p$ being necessary implies that it is
a priori if $p$ is the case. But according to the logic, this is not the case, since $\bigcirc p \rightarrow A(p \rightarrow \bigcirc p)$ is not valid on $\mathcal{M}$. An argument that
this is correct can be found in Edgington (2004, p. 11, attributed to Timothy
Williamson, in conversation).

Logics of necessity often contain $\bigcirc p \rightarrow p$ and $p \rightarrow \bigcirc \neg p$ as theorems, and
these are valid on $\mathcal{M}$ as well. More generally, the logic of $\mathcal{M}$ is a conservative
extension of the strong modal logic $\mathbf{S5}$, in the sense that the formulas valid
on $\mathcal{M}$ that contain no modal operators besides $\bigcirc$ (i.e., neither $\bigcirc \neg$ nor $A$) are
exactly the theorems of $\mathbf{S5}$. The same holds for $A$. Furthermore, the analogous
fact holds for the logic of necessity and actuality of Crossley and Humberstone
(1977) according to real-world validity, as the logic of $\mathcal{M}$ is also a conservative
extension of that logic. These claims can be proven using structural connections
between $\mathcal{M}$ and well-known classes of structures that characterize these logics.

Now that we have a formal logic based on two-dimensional semantics, we have plausible grounds for claiming that essential ideas of two-dimensional
semantics can be put in the form of a systematic combinatorial semantics. We can also apply the logic to the nesting problem. Its central argument can be
written as follows in the formal language, where $N1$ and $N2$ represent the
premises, and $N3$ the conclusion:

$$
\begin{align*}
N1 & \; Ap \rightarrow \bigcirc Ap \\
N2 & \; \bigcirc (Ap \rightarrow p) \\
N3 & \; Ap \rightarrow \bigcirc p
\end{align*}
$$

While the conclusion is a consequence of the premises over $\mathcal{M}$ and the first
premise is valid on $\mathcal{M}$, the second premise and the conclusion are not. That is,
$\{N1, N2\} \vDash_{\mathcal{M}} N3$, $\mathcal{M} \vDash N1$, $\mathcal{M} \not\vDash N2$, and $\mathcal{M} \not\vDash N3$. To see that $N2$ and
$N3$ are not valid on $\mathcal{M}$, consider a model in which $p$ is true throughout the
diagonal, but not throughout the horizontal. So according to the formalization,
proponents of two-dimensional semantics should answer the nesting problem by
denying that all instances of the second premise are true. In particular, they
should claim that if $p$ is a priori and contingent, then it is not necessarily the
case that if $p$ is a priori then $p$ is the case, and thus possible that $p$ is a priori
and $p$ is not the case. This may be surprising, and it may even seem to conflict
with our understanding of apriority. E.g., consider the following sentence, using
parenthesis to settle scope ambiguities:
It is possible that (it is a priori that (grass is green if grass is actually green) and it is not the case that (grass is green if grass is actually green)).

If we propose to solve the nesting problem by rejecting \( \mathcal{N}2 \), then we have to argue that (B) is true, or provide plausible grounds that it is not an instance of a formula like \( \diamond (\mathcal{A} p \land \neg p) \) or \( \diamond (\mathcal{A} (\mathcal{A} p \rightarrow p) \land \neg (\mathcal{A} p \rightarrow p)) \). Both seem quite difficult to do. So although the logic indicates a possible answer to the nesting problem, this may not be the correct answer, or the best answer two-dimensional semantics can offer. E.g., Chalmers (2011, endnote 25) discusses a modification of the analysis of apriority in two-dimensional semantics which would solve the nesting problem differently. I didn’t consider this modification in the formalization because, on the one hand, it makes the semantics more complicated, and on the other hand, the proposal seems to be rather tentative, given that it occurs only in an endnote.

Besides this variant, there are other ways of giving a different formal semantics for the propositional language considered here that still have a claim on capturing the ideas of two-dimensional semantics, since in a number of respects, the informality of the expositions of two-dimensional semantics leaves room for decisions. An example is the treatment of \( \mathcal{O} \). We could differentiate between contexts of utterance and scenarios, and consequently give a semantics using triples instead of pairs. With this, one could define validity in a way that would not count \( \mathcal{O} p \rightarrow p \) as valid, and in general, would give us the logic of necessity and actuality of Crossley and Humberstone (1977) according to general validity, while still counting \( \mathcal{A} p \rightarrow p \) as valid. Therefore, the present development should only be seen as one way of giving a logic based on two-dimensional semantics, although, I hope, a natural and simple one.

### 3 Syntactic Aspects

After formalizing the account of necessity, actuality and apriority given by two-dimensional semantics using the class of matrix \( \text{fwdes} \) \( \text{MD} \) and applying this to the nesting problem, I want to consider some syntactic aspects of the logic of \( \text{MD} \) in this section. In particular, I will provide a complete syntactic characterization of it, and then present a normal form to which all formulas can equivalently be reduced.

As mentioned above, the logic of \( \text{MD} \) is a conservative extension of the logic of necessity and actuality of Crossley and Humberstone (1977) according to real-world validity. As the latter is not normal, neither is the former. In particular, \( \mathcal{G} p \rightarrow p \) is valid on \( \text{MD} \), but \( \Box(\mathcal{G} p \rightarrow p) \) is not. Therefore, standard methods for finding an axiomatization for the logic of a class of frames will not work for the logic of \( \text{MD} \). But we can roughly follow the strategy used, e.g., in Vlach (1973) and Crossley and Humberstone (1977), by first finding an axiomatization of the logic of \( \text{M} \), and syntactically deriving a logic from this, which can then be proven sound and complete with respect to \( \text{MD} \). In doing so, I will make use of some standard definitions and results in modal logic, which can be found, e.g., in Blackburn et al. (2001), whose terminology and notation I largely adopt. First, we show that the logic of \( \text{M} \) is \( 2\text{Dg} \), which is defined as follows:
Definition 3. Let $2Dg$ be the normal modal logic axiomatized by the following formulas:

<table>
<thead>
<tr>
<th>Formula</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>$T \Box \Box \varphi \rightarrow \varphi$</td>
<td>$I_2$ $\Box \varphi \rightarrow \Box \Box \varphi$</td>
</tr>
<tr>
<td>$5 \Diamond \varphi \rightarrow \Box \Diamond \varphi$</td>
<td>$I_4$ $\Diamond \Box \varphi \rightarrow \Box \Diamond \Box \varphi$</td>
</tr>
<tr>
<td>$D_A \Box \varphi \rightarrow \neg \Box \neg \varphi$</td>
<td>$I_3$ $\Box \varphi \rightarrow \Box \varphi$</td>
</tr>
<tr>
<td>$D_c \neg \Box \varphi \rightarrow \Box \neg \varphi$</td>
<td>$I_4$ $A(\Box \varphi \rightarrow \varphi)$</td>
</tr>
</tbody>
</table>

At first glance, this axiomatization consisting of ten formulas might look unwieldy. But the first six axioms are just the axioms of necessity and actuality as found in Davies and Humberstone (1980). (The axiomatization in Crossley and Humberstone (1977) contains an additional axiom, which can be shown to be redundant.) So we only have to add the last four to express the logic of apriority and its interaction with necessity and actuality. The first six axioms are discussed in the texts just mentioned, so I will only say something about the others. $4_A$ and $5_A$ can be understood as saying that apriority has the positive and negative introspection properties. That is, whether it is a priori that $p$ or not, it is a priori whether $p$ is a priori. $I_3$ expresses that if $p$ is a priori then $p$ is actually the case, and $I_4$ says that it is a priori that if $p$ is actually the case, then $p$ is the case.

All of these axioms are Sahlqvist formulas, so the conditions on frames they express can be calculated by the Sahlqvist-van Benthem algorithm. The following table lists the axioms of $2Dg$ and their local frame correspondents, i.e., formulas of predicate logic expressing conditions that are satisfied by a point $w$ of a frame if and only if the corresponding axiom is valid in that frame at that point:

<table>
<thead>
<tr>
<th>Formula</th>
<th>Description</th>
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<tbody>
<tr>
<td>$T \Box \Box \varphi \rightarrow \varphi$</td>
<td>$wR\Box w$</td>
</tr>
<tr>
<td>$5 \Diamond \varphi \rightarrow \Box \Diamond \varphi$</td>
<td>$\forall u((wR\Box u \land wR\Box v) \rightarrow vR\Box u)$</td>
</tr>
<tr>
<td>$D_A \Box \varphi \rightarrow \neg \Box \neg \varphi$</td>
<td>$\exists v(wRa v)$</td>
</tr>
<tr>
<td>$D_c \neg \Box \varphi \rightarrow \Box \neg \varphi$</td>
<td>$\forall u((wR\Box a u \land wRa u) \rightarrow v = u)$</td>
</tr>
<tr>
<td>$I_1 \Box \varphi \rightarrow \Box \varphi$</td>
<td>$\forall v(wR\Box v \rightarrow wR\Box v)$</td>
</tr>
<tr>
<td>$I_2 \Box \varphi \rightarrow \Box \varphi$</td>
<td>$\forall u((wR\Box u \land vRa u) \rightarrow wRa u)$</td>
</tr>
<tr>
<td>$4_A \Box \varphi \rightarrow \Box \Box \Box \varphi$</td>
<td>$\forall u((wR\Box u \land wRa u) \rightarrow wRa u)$</td>
</tr>
<tr>
<td>$5_A \Box \varphi \rightarrow \Box \Box \Box \varphi$</td>
<td>$\forall u((wR\Box u \land vRa u) \rightarrow vRa u)$</td>
</tr>
<tr>
<td>$I_3 \Box \varphi \rightarrow \Box \varphi$</td>
<td>$\forall v(wR\Box v \rightarrow wR\Box v)$</td>
</tr>
<tr>
<td>$I_4 \Box \varphi \rightarrow \Box \varphi$</td>
<td>$\forall v(wR\Box v \rightarrow wR\Box v)$</td>
</tr>
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</table>

Furthermore, the fact that the axioms are Sahlqvist formulas implies that $2Dg$ is strongly complete with respect to $Fr_{2Dg}$, the class of $2Dg$-frames (frames in which $2Dg$ is valid):

**Theorem 4.** $2Dg$ is sound and strongly complete with respect to $Fr_{2Dg}$.

**Proof.** By the Sahlqvist completeness theorem; see, e.g., Blackburn et al. (2001, Theorem 4.42).
satisfiable on $M$. These claims will be established by proving that $R$ is the class of point-generated subframes of $2Dg$-frames, as well as the class of bounded morphic images of matrix frames. The desired claims about satisfiability follow from these structural connections by well-known invariance results. Since a logic is sound and strongly complete with respect to a class of frames if and only if every set of formulas is consistent in the former if and only if it is satisfiable on the latter, the completeness of $2Dg$ with respect to $M$ follows by Theorem 4. In a slightly different form, the intermediate class of frames $R$ is used in Restall (2011), so I will call them Restall frames. Calling a relation a function if it is serial and functional, and writing $\text{im}(R)$ for the image of a relation $R$, they can be defined as follows:

**Definition 5.** A Restall frame is a frame $\mathfrak{F} = \langle W, R_\Box, R_\@, R_A \rangle$ such that

- $R_\Box$ is an equivalence relation,
- $R_\@$ is a function that maps any two $R_\Box$-related points to the same point, which is $R_\Box$-related to both of them, and
- $wR_Av$ iff $v \in \text{im}(R_\@)$.

Let $R$ be the class of Restall frames.

**Lemma 6.** Every Restall frame is a point-generated subframe of a $2Dg$-frame.

*Proof.* Consider any Restall frame $\mathfrak{F} = \langle W, R_\Box, R_\@, R_A \rangle$. We first show that for any $w \in W$, $\mathfrak{F}_w$ (the subframe of $\mathfrak{F}$ generated by $w$) is $\mathfrak{F}$ itself. Consider any $v \in W$. Since $R_\@$ is serial, there is a $u \in W$ such that $vR_\@u$. So also $vR_\Box u$, and by symmetry of $R_\Box$, $uR_\Box v$. It is also the case that $u \in \text{im}(R_\@)$, so $wR_Au$. It follows from $wR_Au$ and $uR_\Box v$ that $v$ is in $\mathfrak{F}_w$. As $v$ was chosen arbitrarily, $\mathfrak{F}_w = \mathfrak{F}$.

To show that $\mathfrak{F}$ is itself a $2Dg$-frame, it suffices to go through the axioms of $2Dg$ and verify that the properties defined by them are satisfied by Restall frames. This is straightforward for all axioms except $I4$. For this, we can reason as follows: let $v \in \text{im}(R_A)$. Then $v \in \text{im}(R_\@)$, so there is a $u$ such that $uR_\@w$. It follows that $uR_\Box v$, and therefore that $R_\@$ must map $u$ and $v$ to the same point. So $vR_Av$, which means that $R_\@$ is reflexive on $\text{im}(R_A)$.

The next result will make use of the fact that $N1$, the formula representing the first premise of the nesting argument, is a theorem of $2Dg$. This is shown in the following lemma:

**Lemma 7.** $\vdash_{2Dg} Ap \to \Box Ap$.

*Proof.* By the following derivation:
Note that $Ap \to \Box Ap$ is also a Sahlqvist formula, and that it is therefore straightforward to calculate that it has the following local frame correspondent: $\forall w(u(wRw \land vRau) \to wRau)$.

**Lemma 8.** Every point-generated subframe of a $2Dg$-frame is a Restall frame.

*Proof.* Consider any $2Dg$-frame $\mathfrak{F} = \langle W, R_\Box, \Box R, R_A \rangle$ and $w \in W$. Let $\mathfrak{F}_w = \langle W', R_w', R_v', R_u' \rangle$ be the subframe generated by $w$. Since validity is preserved under taking generated subframes, all of the axioms of $2Dg$ are valid in $\mathfrak{F}_w$.

Using $T_\Box$ and $S_\Box$, it is routine to show that $R_\Box$ is an equivalence relation. Likewise, $D_\Box$ and $D_\Box$ imply that $R_\Box$ is a function. With this, it follows from $I1$ and $I2$ that $R_\Box$ maps $R_\Box$-related points to the same point, to which both are $R_\Box$-related.

To show that $vR_A'u$ if and only if $w \in \text{im}(R_\Box)$, assume first that $vR_A'u$. Then by $I4$, $wR_\Box'u$, and so $u \in \text{im}(R_\Box)$. It only remains to show that if $u \in \text{im}(R_\Box)$, then $vR_A'u$. We will do this in the rest of this proof, adopting the notation to write $R[Y]$ for the image of a set $Y$ under a relation $R$. Let $w'$ be the element of $W'$ such that $wR_\Box'w'$. The existence and uniqueness of this point are guaranteed by the fact that $R_\Box$ is a function. We first prove a preliminary claim:

**Claim 1:** $W' = X$, where $X = \{vR_\Box[R_A\{w]\}]\}$. Clearly $X \subseteq W'$. We first show that $w \in X$, and then that $X$ is closed under each of the relations, that is, that $R_\Box[X] \subseteq X$ for every modality $\Box$.

$wR_\Box'w$, so both $wR_\Box w'$ and $wR_\Box w$. Since $R_\Box$ is symmetric, $w'R_\Box w$ and therefore $w \in X$. Assume that $v \in R_\Box[X]$. Then there is a $u \in X$ such that $uR_\Box v$. Since $u \in X$, there is a $u' \in W$ such that $wR_\Box u'$ and $u'R_\Box v$. By transitivity of $R_\Box$, $u'R_\Box v$, and so $v \in X$. Assume that $v \in R_\Box[X]$. Then there is a $u \in X$ such that $uR_\Box v$. By $I1$, $R_\Box \subseteq R_\Box$, so $uR_\Box v$. That $v \in X$ follows by transitivity of $R_\Box$ as before. Assume that $v \in R_\Box[X]$. Then there is a $u \in X$ such that $uR_\Box v$, and therefore a $u' \in W$ such that $wR_\Box u'$ and $u'R_\Box v$. By $\Box$, follows that $wR_\Box v$, and so by transitivity of $R_\Box$ that $wR_\Box v$. Since $R_\Box$ is reflexive, $v \in X$. This concludes the proof of claim 1.

Now consider any $u \in \text{im}(R_\Box)$ and $v \in W'$. We have to prove that $vR_A'u$. We do this by first proving that $vR_A'w'$ and then that $w'R_A'u$.

**Claim 2:** $vR_A'w'$. Since $v \in W'$, it follows from claim 1 that there is a $w' \in W'$ such that $wR_A'v$ and $vR_A'w$'. By symmetry of $R_A'$, $vR_A'D'$. Since $wR_\Box'w'$, by $I3$ also $wR_A'w'$. So since $R_A'$ is euclidean, $vR_A'w'$'. By $\Box$, it follows that $vR_A'w'$. 

| (1) | $Cp \to ACp$ | $5_A$ |
| (2) | $ACp \to @Cp$ | $I3$ |
| (3) | $Cp \to @Cp$ | $(1), (2)$ |
| (4) | $\neg@\neg \neg \neg p \to A \neg \neg p$ | $(3)$ |
| (5) | $@Ap \to Ap$ | $\Box _\Box$ $(4)$ |
| (6) | $\Box \Box Ap \to \Box Ap$ | $K_\Box$ $(5)$ |
| (7) | $\Box Ap \to \Box Ap$ | $I2$ |
| (8) | $\Box Ap \to \Box Ap$ | $(6), (7)$ |
| (9) | $Ap \to Ap$ | $4_A$ |
| (10) | $\Box Ap \to Ap$ | $I3$ |
| (11) | $Ap \to Ap$ | $(9), (10)$ |
| (12) | $Ap \to Ap$ | $(8), (11)$ |
Claim 3: \( w'R_A^u \). Since \( u \in W' \), there is a \( u' \in W' \) such that \( w'R_A^u u' \) and \( u' R_A^u u \). As we’ve seen before, \( w'R_A^u u' \), so since \( R_A^u \) is euclidean, \( u'R_A^u u' \). Also \( u \in \text{im}(R_A^u) \), so there is a \( u'' \in W' \) such that \( u'' R_A^u u \). By 11 also \( u'' R_A^u u \), and with the fact that \( R_A^u \) is an equivalence relation, \( u'R_A^u u'' \). So by 12, it follows that \( u'R_A^u u \), and with 13, \( u'R_A^u u \). Since \( R_A^u \) is transitive, \( u'R_A^u u \).

By transitivity of \( R_A^u \), it follows from claims 2 and 3 that \( v'R_A^u u \).

With these Lemmas, it follows that \( 2Dg \) is sound and complete with respect to \( R \). Without much effort, we could use Restall (2011, Theorem 8) to conclude that \( 2Dg \) is sound and complete with respect to \( M \). But in order to bring out the structural connections between the classes of frames, we continue with our initial proof strategy:

Lemma 9. Every bounded morphic image of a matrix frame is a Restall frame.

Proof. By checking the conditions on Restall frames, one can verify that matrix frames are Restall frames. With this, the claim follows from the fact that \( R \) is closed under taking bounded morphic images, which is routine to prove.

Lemma 10. Every Restall frame is a bounded morphic image of a matrix frame.

Proof. Let \( \mathcal{F} = < W, R, R_A, R_\mathfrak{A} > \) be a Restall frame. We proceed by constructing a matrix frame \( \mathcal{F}' \) and a surjective bounded morphism \( f \) from \( \mathcal{F}' \) to \( \mathcal{F} \). I will use the following notation: \( [x]_E \) is the equivalence class of \( x \) under the equivalence relation \( E \). For a relation \( R \) that is a function, \( R(x) \) is the unique \( y \) such that \( x R y \).

Let \( f \) be a set of cardinality \( |W| \) and \( \mathcal{F}' \) the matrix frame based on \( I \). Let \( \alpha : I \rightarrow W \) be a surjection, and for every \( i \in I \), let \( \beta_i : I \rightarrow [\alpha(i)]_{R_\mathfrak{A}} \) be a surjection such that \( \beta_i(i) = R_\mathfrak{A}(\alpha(i)) \). Such surjections exist for cardinality reasons, and the fact that \( R_\mathfrak{A} \) is a function for which \( R_\mathfrak{A} \subseteq R_\mathfrak{C} \) holds. We define \( f : I \times I \rightarrow W \) by \( f((i, j)) = \beta_j(i) \).

We can prove that \( f \) is a surjective bounded morphism from \( \mathcal{F}' \) to \( \mathcal{F} \). To show that it is a bounded morphism, one only has to go through the modalities and check the forth and back conditions. For surjectivity, consider any \( w \in W \). For some \( i \in I \), \( \alpha(i) = w \). So \( \beta_i : I \rightarrow [w]_{R_\mathfrak{C}} \) is a surjective function. Therefore, there is a \( j \in I \) such that \( \beta_j(i) = w \). So \( f((j, i)) = w \).

Theorem 11. \( 2Dg \) is sound and strongly complete with respect to \( M \).

Proof. The preceding lemmas establish that Restall frames are both the point-generated subframes of \( 2Dg \)-frames as well as the bounded morphic images of matrix frames. Since truth is invariant under taking generated submodels (see Blackburn et al. (2001, Proposition 2.6)) as well as bounded morphisms between models (see Blackburn et al. (2001, Proposition 2.14)), one can show with standard arguments that a set of formulas is satisfiable on \( Fr_{2Dg} \) if and only if it is satisfiable on \( R \), and that this is the case if and only if it is satisfiable on \( M \). With Theorem 4, the claim follows.

In the same way in which Crossley and Humberstone (1977) derive their logic of real-world validity from their logic of general validity, we can use \( 2Dg \) to define a logic \( 2D \), and infer from Theorem 11 that it is sound and complete with respect to \( M_D \). Since \( M_D \) is the formal semantics that captures two-dimensional semantics, this completeness result means that the following definition of \( 2D \) gives us a syntactic characterization of our logic of two-dimensional semantics:
Definition 12. \( \vdash_{2D} \varphi \) if and only if \( \vdash_{2Dg} \Diamond \varphi \).

Theorem 13. 2D is sound and strongly complete with respect to MD.

Proof. We show that any set of formulas is 2D-consistent if and only if it is satisfiable on MD. Note that it is straightforward to verify that 2D is quasi-normal. First, let \( \Gamma \) be a 2D-inconsistent set. Then there are \( \varphi_1, \ldots, \varphi_n \in \Gamma \) such that \( \vdash_{2D} \neg \bigwedge_{i \leq n} \varphi_i \). So by definition of 2D, \( \vdash_{2Dg} \neg \bigwedge_{i \leq n} \varphi_i \). Consider any matrix FWDE \( \mathcal{F} \) with set of points \( W \), relation \( R_\alpha \), and distinguished points \( D \), and let \( w \in D \). By the soundness of 2Dg, \( \mathcal{F}, w \vdash \neg \bigwedge_{i \leq n} \varphi_i \). Since \( wR_\alpha w \), also \( \mathcal{F}, w \vdash \neg \bigwedge_{i \leq n} \varphi_i \). Hence \( \Gamma \) is not satisfiable on MD.

Now, let \( \Gamma \) be a set that is not satisfiable on MD. Assume for contradiction that \( \Gamma^\sharp = \{ \Box \varphi : \varphi \in \Gamma \} \) is satisfiable on \( M \). Then there is a matrix frame \( \mathfrak{F} \) with set of points \( W \) and relation \( R_\alpha \), and a point \( w \in W \) such that \( \Gamma^\sharp \) is satisfiable in \( \mathfrak{F} \) at \( w \). Since \( R_\alpha \) is a function, there is a \( v \in W \) such that \( wR_\alpha v \), so \( \Gamma \) is satisfiable in \( \mathfrak{F} \) at \( v \). But then \( v \) is a diagonal point, so \( \Gamma \) is satisfiable on MD. \( \mathfrak{F} \), \( \Gamma^\sharp \) is not satisfiable on M. By strong completeness of 2Dg it follows that \( \Gamma^\sharp \) is 2Dg-inconsistent, and so that there are \( \varphi_1, \ldots, \varphi_n \in \Gamma \) such that \( \vdash_{2Dg} \neg \bigwedge_{i \leq n} \varphi_i \). Since \( D_\alpha \) and \( D_\beta \) are theorems of 2Dg, \( \Box \) distributes over Boolean connectives in 2Dg. Therefore \( \vdash_{2Dg} \neg \bigwedge_{i \leq n} \varphi_i \), and so by definition of 2D, \( \vdash_{2D} \neg \bigwedge_{i \leq n} \varphi_i \). Hence \( \Gamma \) is 2D-inconsistent.

This concludes the axiomatization of the logic of MD. We can now derive as a corollary that the philosophically implausible simplification of identifying scenarios and worlds in matrix frames and FWDES does not affect the logic of the language considered here. As noted above, it would be more adequate if matrix frames were generalized to look like tables that contain at least as many rows as columns and determine a function that maps every row (representing a scenario) to a column (representing its possible world). More formally, we can define:

Definition 14. A rectangular frame is a frame \( \mathfrak{F} = (W, R_\Box, R_\alpha, R_A) \), such that \( W = P \times S \) for some sets \( P \) (representing possible worlds) and \( S \) (representing scenarios), there is a surjective function \( d : S \to P \) (representing the function that maps scenarios to their possible worlds), and the relations are given by the following conditions:

- \( (x, y)R_\Box(x', y') \) iff \( y = y' \)
- \( (x, y)R_\alpha(x', y') \) iff \( y = y' \) and \( x' = d(y') \)
- \( (x, y)R_A(x', y') \) iff \( x' = d(y') \)

A rectangular FWDE is an FWDE based on a rectangular frame such that the set of distinguished elements \( D = \{ (d(x), x) : x \in S \} \). Let \( \text{Rec} \) be the class of rectangular frames and \( \text{RecD} \) the class of rectangular FWDES.

Corollary 15. 2D is sound and strongly complete with respect to \( \text{RecD} \).

Proof. Every matrix frame is a rectangular frame and every rectangular frame is a Restall frame. Since a set of formulas is satisfiable on \( M \) if and only if it is satisfiable on \( R \), it follows that it is satisfiable on \( M \) if and only if it is satisfiable on \( \text{Rec} \), and therefore with Theorem 11 that 2Dg is sound and strongly complete with respect to \( \text{Rec} \). Analogously to the proof of Theorem 13, it can be inferred that 2D is sound and strongly complete with respect to \( \text{RecD} \). \( \square \)
So on the level of the propositional logic of necessity, actuality and apriority, the simplification of not distinguishing between scenarios and worlds made in matrix frames is not problematic. One might also wonder whether there could be worlds that are not the world of any scenario, which would correspond to dropping the requirement for $d$ to be surjective in the definition of rectangular frames, or whether the space of worlds could vary with the scenario (see Chalmers (2004, pp. 213–214) for motivation), which would also result in a class of frames properly containing $\text{Rec}$. But again, these classes of frames would be contained in $\mathbb{R}$, and the equivalence of the logics would follow as above. It should be noted that these observations do not in general extend to languages containing additional operators, e.g., propositional quantifiers or the so-called fixedly operator used in Davies and Humberstone (1980). See also Humberstone (2004, p. 20) and Restall (2011, p. 12) for similar observations.

Besides giving a syntactic characterization of the logic of $\text{MD}$ as we have done with $\text{2D}$, we can also show that every formula can equivalently be written in a certain syntactically simple form. That is, we can prove that any formula $\varphi$ is $\text{2D}$-equivalent to a formula $\psi$ of this form, where two formulas are $\text{2D}$-equivalent if $\varphi \leftrightarrow \psi$ is a theorem of $\text{2D}$. This is an extension of a familiar result about the logic $\text{S5}$, for which we can prove that every formula is equivalent to a formula that contains no nested modal operators. More specifically, let a formula be in $\text{S5}_{\Box}$-CNF (CNF stands for conjunctive normal form) if it is a finite conjunction of finite disjunctions of formulas of the form $\Box \varphi$, $\Diamond \varphi$ or $\varphi$, where $\varphi$ contains no modal operators. We can then prove that every formula (in the language containing only the modality $\Box$) is $\text{S5}_{\Box}$-equivalent to one in $\text{S5}_{\Box}$-CNF: see Hughes and Cresswell (1996, p. 101). The analog to containing no nested operators in the context of $\text{2D}$ is to contain only $\Box$ and $A$, and these only unnestedly or with $\Box$ nested in $A$. More precisely, we can define the following normal form:

**Definition 16.** A formula is in $\text{2D}$-CNF if it is a finite conjunction of finite disjunctions of formulas of the form $A \varphi$, $C \varphi$ or $\varphi$, where $\varphi$ is in $\text{S5}_{\Box}$-CNF.

**Theorem 17.** Every formula is $\text{2D}$-equivalent to one in $\text{2D}$-CNF.

**Proof.** Define a formula to be in $\text{2Dg}$-CNF if it is a finite conjunction of finite disjunctions of formulas of the form $A \varphi$, $C \varphi$ or $\psi$, where $\varphi$ is in $\text{S5}_{\Box}$-CNF and $\psi$ is in $\text{S5}_{\Box}$-CNF or of the form $\theta \chi$, where $\chi$ contains no modal operators. We can prove that every formulas is $\text{2Dg}$-equivalent to one in $\text{2Dg}$-CNF by an induction on the complexity of formulas analogously to the case of $\text{S5}_{\Box}$-CNF. Any formula in $\text{2Dg}$-CNF contains $\theta$ only unnestedly. The claim follows, as removing any such occurrence of $\theta$ produces a $\text{2D}$-equivalent formula in $\text{2D}$-CNF.

Since no formula in $\text{2D}$-CNF contains an occurrence of $\theta$, this also shows that the actuality operator is redundant in $\text{2D}$ in the sense that any formula is equivalent to one not containing $\theta$, which was already shown in Crossley and Humberstone (1977) and Hazen (1978) for their logic of necessity and actuality according to real-world validity:

**Corollary 18.** For every formula, there is a $\text{2D}$-equivalent formula not containing $\theta$. 

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4 Conclusion

The central claims of two-dimensional semantics about meaning using primary and secondary intensions and their relations to necessity and apriority as well as their interactions with actuality can be systematized using standard ways of providing a combinatorial semantics. Here, this was done using a propositional modal language and a formal semantics based on a class of Kripke frames with distinguished elements. Applying this logic to the nesting problem indicates one way of answering it. The axiomatization of the logic shows that it can be given a sound and complete proof system, which provides a way of testing two-dimensional semantics using its claims concerning the relations and interactions of necessity, actuality and apriority.

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References


