Valuation of Basket Credit Derivatives in Structural Jump-Diffusion Models

Karolina Bujok\(^\ast\) and Christoph Reisinger\(^\dagger\)

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Abstract

We consider a multi-dimensional structural credit model, where each company’s asset value follows a jump-diffusion process, and is connected with other companies via global factors. Based on ideas in [5], where the joint density of a portfolio is evolved in a large basket approximation, we develop an algorithm for the efficient estimation of CDO index and tranche spreads, based on a semi-analytic solution of the resulting SPDE in combination with finite differences. We verify the validity of this approximation numerically by comparison to results obtained by direct Monte Carlo simulation of the basket constituents. A calibration exercise demonstrates that the model is flexible enough to match CDO spreads from pre-crisis and crisis periods, giving estimated parameters with economically convincing values. We suggest extensions for the use of the model in practice.

1 Introduction

Despite a dip in popularity, collateralized debt obligations (CDO) are useful financial instruments, which help manage the credit risk of a portfolio of defaultable assets. What the credit crunch of 2008 did show is that complex credit instruments, like CDOs, require a more sophisticated, and, most importantly, robust pricing and hedging methodology.

Most currently used models fall into one of two categories, reduced-form and structural. In the first approach, a random time of default is modelled directly, typically as a jump of a Poisson process. Due to simplicity and ease of implementation, reduced-form models are commonly used in practice, even though they are detached from any economic grounding. For a survey of reduced-form models see [29] and [1].

An alternative are structural models, which are based on the natural observation that default depends on the value of a company’s assets. In these models, default happens when the asset value falls below a given barrier. Merton, in the original work [24], assumes that a firm’s asset value follows geometric Brownian motion, and that the company can default only at the maturity of the debt, namely if the assets’ value is lower than the debt. Using first passage time theory, Black and Cox [2] extended the model by allowing default at any time.

\(^\ast\)Mathematical Institute, University of Oxford, 24-29 St Giles, Oxford OX1 3LB, UK. E-mail: bujok@maths.ox.ac.uk

\(^\dagger\)Mathematical Institute, University of Oxford, 24-29 St Giles, Oxford OX1 3LB, UK. E-mail: reisinge@maths.ox.ac.uk
As Lipton and Sepp [23] note, when Credit Default Swaps (CDSs) became liquid in the early 2000s, it also became clear that diffusion models cannot explain high short-term CDS spreads observed in the market. The reason behind this is that in diffusion models default is a predictable event, hence the probability of default of a healthy firm in a short time is close to zero. Zhou [33] adds that under diffusion models, the term structure of credit spreads of a non-distressed firm always starts at zero and then increases, which is not always the case in the market, where credit spread curves can be flat or even downward-sloping. One of the solutions to this problem suggested in the literature is to model a company’s asset value process by a jump-diffusion ([33], [14], [15], [19], [23]) or more general Lévy process [9].

In a jump-diffusion model, a default can happen both expectedly, due to the diffusion part and unexpectedly, due to the jump part, by a sudden fall in a company’s value, which is an intuitive way to think about defaults. Kiesel and Scherer [19] show that in a jump-diffusion model the limit of CDS spreads at maturity is positive, unlike in the diffusion case, such that a better fit to the market data can be achieved.

In order to price basket credit derivatives, like CDOs, the joint default time distribution of firms in a basket is needed. Again, there are two main approaches to modelling this joint distribution, top-down and bottom-up. In the first approach, the joint default distribution of a whole portfolio is modelled directly. As Schönbucher argues, [30], it is natural to model directly the dynamics of a portfolio, when the price of a portfolio derivative is needed. Although some individual information of entities is lost, such models can often better match observed spreads. For examples of the top-down approach see [30] and references therein.

In the bottom-up approach, individual entities are analysed and then linked together to obtain a joint default distribution. Copula models and conditionally independent factor (CIF) models are examples of the bottom-up approach. Under copula models, a joint default density is constructed from individual default distributions of entities and an assumed dependence structure via copula functions (see [29] for a survey of copula modelling). Under CIF models, entities are linked to each other via a set of common factors. Conditional on realisation of the factors, defaults of entities are independent, which can be used to construct the joint default distribution (for a survey of CIF models see [29]). As noted in [5], a drawback of both models is that they are static, hence cannot be used for pricing dynamic credit derivatives, like forward starting CDOs.

Another example of the bottom-up approach are multi-name structural models. The models are by construction dynamic, hence can be used for pricing exotic credit derivatives. Published results on multi-dimensional structural models are sparse, since increasing dimensionality decreases their tractability and increases computational time dramatically. For a first passage time model in a diffusion setting, Zhou [34] derives analytical solutions for a portfolio consisting of two companies; Hull et al. [16] suggest a Monte Carlo algorithm when a large portfolio is considered. In order to price a CDO without simulating asset value paths for each company, Bush et al. [5] suggest to approximate the joint density of the asset portfolio by the large basket limit of the empirical measure of asset values in the basket. The joint density is then given by the solution to a Stochastic Partial Differential Equation (SPDE).

In this article, we develop a multi-dimensional structural credit model, where each company follows a jump-diffusion process with log-normally distributed jumps, and is connected with other companies via global factors. In order to calculate the portfolio loss
distribution, we build on ideas developed in [5] and [17]. The contribution of this work is to construct a particular multi-name jump-diffusion model, to develop an efficient numerical implementation and calibration, and to provide an analysis of the results implied by the model.

The article is organised as follows. In Section 2, we briefly explain the credit instruments that we analyse later. In Sections 3 and 4, we present our structural model for single- and multi-name credit derivatives and derive an SPDE model for the joint density of a large credit portfolio. Section 5 gives details of the numerical implementation of the model. Finally, in Section 6, we outline how to calibrate the model efficiently and analyse if the model is flexible enough to match market spreads both in pre-crisis in and crisis periods, while Section 7 concludes with remarks on further improvements and the practical use of the model.

2 Basket Credit Derivatives

A collateralized debt obligation (CDO) is a set of fixed income securities whose payments depend on credit events in a pool of defaultable assets, like loans, bonds or credit default swaps (CDSs). In this article, we analyse a synthetic CDO, whose portfolio consists of CDSs.

A CDO offers products with different risk profiles, based on so-called tranches. The risk connected to tranches varies from equity tranches, which are typically unrated or speculative-grade, investment-grade mezzanine tranches, to senior tranches, which can even have AAA rating. By dividing the pool of defaultable assets into tranches, for instance investors who would not be allowed to invest directly in the underlying assets, because of their too low ratings, can invest in senior tranches.

Each tranche of a CDO is defined by an attachment point, \( a \), and a detachment point, \( d > a \), which are typically given as a percentage of a portfolio notional, e.g. for the equity tranche typically \( a = 0\% \) and \( d = 3\% \). The tranche notional is given as the difference between \( a \) and \( d \).

Let the reference portfolio for a synthetic CDO consist of \( N \) CDSs, each with notional \( N_0 \). The total loss of the portfolio is given by

\[
L_t = N_0(1 - R) \sum_{i=1}^{N} 1\{\tau_i \leq t\},
\]

where \( R \) is a recovery rate, \( \tau_i \) is the default time of the \( i \)-th entity. The outstanding notional, \( Z_t \), of a single tranche is given by

\[
Z_t = [d - L_t]^+ - [a - L_t]^+.
\]

The STCDO (Single Tranche CDO) protection buyer pays a running spread \( c \) on the outstanding tranche notional only, at a set of payments dates \( T_j \), where \( 1 \leq j \leq n \). Denote the payment intervals by \( \varrho_j = T_j - T_{j-1} \), and the value of a bank account at time \( t \) by \( b_t \).

We now derive a formula for \( c \), following [5], and assume the existence of a pricing measure \( \mathbb{Q} \) to be discussed later. For an overview of pricing approaches see e.g. [29], [3].
The value of the fee leg of the contract, \(B\), is then given by
\[
cB = c \sum_{j=1}^{n} \frac{\theta_j}{b_{T_j}} \mathbb{E}^Q[Z_{T_j}],
\]
The value of the protection leg, \(D\), is a function of tranche losses incurred in the intervals \((T_{j-1}, T_j]\), and can be written as
\[
D = \sum_{j=1}^{n} \frac{1}{b_{T_j}} \mathbb{E}^Q[Z_{T_j-1} - Z_{T_j}].
\]
Since at time \(t = 0\) the value of fee and protection legs are equal, \(cB = D\), such that the STCDO spread is given by
\[
c = \frac{D}{B}.
\]
Similarly, for a synthetic CDO index, the pricing formula is
\[
c^I = \frac{(1 - R) \sum_{j=1}^{n} \frac{1}{b_{T_j}} \mathbb{E}^Q[Z_{T_j-1}^I - Z_{T_j}^I]}{\sum_{j=1}^{n} \frac{\psi_j}{b_{T_j}} \mathbb{E}^Q[Z_{T_j}^I]},
\]
where the outstanding index notional is given as
\[
Z_t^I = N_0 \sum_{i=1}^{N} \mathbf{1}_{\{\tau_i > t\}}.
\]
For a single-name CDS, we have
\[
c^{CDS} = \frac{(1 - R) \sum_{j=1}^{n} \frac{1}{b_{T_j}} \mathbb{E}^Q[1_{\{T_{j-1} > \tau\}} - 1_{\{T_j > \tau\}}]}{\sum_{j=1}^{n} \frac{\psi_j}{b_{T_j}} \mathbb{E}^Q[1_{\{\tau > t\}}]},
\]
where \(\mathbb{E}^Q[1_{\{\tau > t\}}] = P(\tau > t)\) is the survival probability of the reference entity.
Tranche spreads depend crucially on the joint default probability of CDSs and hence on the dependence structure of the underlying firms. In contrast, index spreads only depend on expected losses in the entire portfolio.

3 A Single-Name Structural Model

We begin by discussing a model and pricing formulae for the individual CDSs in this section, and present the full multi-name model in Section 4.

3.1 The Model Setup
Let \(A_t\) be the company’s asset value whose process is given by
\[
\frac{dA_t}{A_t} = \alpha \, dt + \sigma \, dW_t + (Y - 1) \, dN_t, \tag{3.1}
\]
where $\sigma$ is the asset volatility, $W$ is a standard Brownian motion, $N$ a Poisson process with intensity $\lambda$, $Y_i$ is the size of the $i$th jump of the compound Poisson process $Z_t = \sum_{i=0}^{N_t} (Y_i - 1)$. We assume that $W$, $N$, $Y$ are mutually independent. The random variables $\{Y_i\}$ are independent and identically distributed (i.i.d.) with $\nu = \mathbb{E}^Q[Y_i - 1]$.

We assume that we are able to hedge default risk and work under the risk-neutral pricing measure $Q$, such that the instantaneous expected rate of change of the company’s assets equals the interest rate $r$, $\mathbb{E}^Q[dA_t/A_t] = rdA$. Since $\mathbb{E}^Q[(Y - 1)dN_t] = \lambda \nu dt$, $\alpha = r - \lambda \nu$.

A solution to (3.1) is given by

$$A_t = A_0 \exp \left\{ \left( \alpha - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right\} \prod_{i=0}^{N_t} Y_i, \quad A_0 > 0, \quad Y_0 = 1.$$ (3.2)

If $\lambda = 0$, then $N_t = 0$, and we obtain geometric Brownian motion. As Merton [25] notes, if we ignore the continuous part it is easy to see that $Y - 1$ is the relative change of $A_t$. What is more, in the special case when $\{Y_i\}$ is log-normally distributed, $A_t$ is also log-normally distributed.

Following Black and Cox [2], one can define the default time $\tau$ as the first passage time of the company’s asset value of a constant default barrier $B$,

$$\tau = \inf \{ t > 0 : A_t \leq B \}.$$ (3.3)

In order to eliminate explicit dependence on $B$, we introduce the distance-to-default

$$X_t = \frac{1}{\sigma} (\ln(A_t) - \ln(B)).$$

By applying Itô’s lemma to (3.3) and using (3.1), we obtain

$$dX_t = \frac{1}{\sigma} (r - \lambda \nu - \frac{1}{2} \sigma^2) dt + dW_t + \Pi dN_t,$$ (3.4)

where $\Pi = \ln Y/\sigma$. The first passage time $\tau$ of $X_t$ is now

$$\tau = \inf \{ t > 0 : X_t \leq 0 \}.$$ (3.3)

Implicit here was the assumption that default can be monitored continuously, while we will later work in a framework where default can only be detected at a discrete set of times $t_1, \ldots, t_n$, and then

$$\tau = \inf \{ t \in \{t_1, \ldots, t_n\} : X_t \leq 0 \}.$$ (3.3)

3.2 Distribution of the Jump Amplitude

The aim of this article is to develop a computationally tractable, while economically convincing structural default model. To meet these conditions, we are looking for a distribution of jump amplitudes that enables sufficiently fast calculation of the survival probability of individual firms, yet gives realistic market dynamics.

Zhou [33] suggests a jump-diffusion model with log-normal jumps, as first introduced in the general context of financial modelling by Merton [25] as an extension to the Black-Scholes approach. In contrast, Kou [20], Hilberink and Rogers [14], Hu and Ye [15],
Lipton and Sepp [23] propose a double exponential jump size distribution. Kou [20] argues that although both types of jumps can lead to the leptokurtic feature of equity returns observed in the market, in continuous time double-exponential jump-diffusion models have better analytical tractability. As Kou and Wang [21] show, due to the lack of memory associated with the exponential distribution, there exists an analytical solution, via Laplace transforms, to the distribution of first passage times for the double exponential case. Such a solution does not exists for the log-normal model. However, as Ramezani and Zeng [27] note, the transition density of log-normal jump-diffusions has a more convenient form than of double-exponential ones.

What is more, [27] assess empirically the performance of double exponential jump-diffusion compared to log-normal jump-diffusion and geometric Brownian motion in matching stock market prices. They find that both double-exponential and log-normal jump-diffusions give better fit to market data than geometric Brownian motion. In their study, double-exponential jump-diffusions outperform log-normal ones when stock indices are concerned, however, for individual stocks the results are inconclusive. It should be noted that a slightly better performance of the double-exponential jump-diffusion model can be attributed to the fact that in the model there is one more parameter to fit compared to the log-normal case.

We focus in the following on the log-normal case, for reasons of overall tractability which will become clear later.

3.3 Computing Survival Probabilities for Log-Normal Jumps

A central role in the pricing of CDSs, which are used for the calibration of CDO models in our framework, plays the survival probability $P(\tau > t)$. As for a log-normal jump-diffusion model an analytical formula for the survival probability does not exist, Zhou [33] applies a Monte Carlo algorithm to calculate survival probabilities. Since this approach is computationally demanding when a portfolio of CDSs is used to price CDO, in order to obtain an analytical approximation to survival probabilities, Willemann [32] assumes that a firm has survived to a given monitoring time if at this particular time the asset value is above the barrier. As the author notes, this is a crude approximation, and the survival probability in the model is overestimated, especially when $t$ is high.

If defaults are monitored at a discrete set of times, $t_1, \ldots, t_n$, as we will assume in the following,

$$P(\tau > t_n) = P(X_{t_n} > 0, X_{t_{n-1}} > 0, \ldots, X_{t_1} > 0)$$

where $P^W(\cdot)$ denotes the survival probability in Willemann’s model.

We now derive a practically tractable algorithm to calculate the survival probability for log-normally distributed jumps.

**Proposition 3.1.** Let $0 < t_n \leq T$ be a monitoring date, $X$ a jump-diffusion process driven by (3.4). The survival probability at time $t_n$, for $n \geq 2$, is given recursively by
\( p(X_{tn} = x_{tn}, \tau > t_{n-1}) = \int_0^\infty p(X_{tn} = x_{tn} \mid X_{tn-1} = x_{tn-1}) p(X_{tn-1} = x_{tn-1}, \tau > t_{n-2}) \, dx_{tn-1}, \)

\[ P(\tau > t_n) = \int_0^\infty p(X_{tn} = x_{tn}, \tau > t_{n-1}) \, dx_{tn}. \quad (3.5) \]

**Proposition 3.2.** The density of \( X_{tn} \), conditional on \( X_{tn-1} \), for \( n \geq 1 \), is given by

\[ p(X_{tn} = x_{tn} \mid X_{tn-1} = x_{tn-1}) = \sum_{c=0}^\infty p(X_{tn} = x_{tn} \mid X_{tn-1} = x_{tn-1}, \Delta N_{tn} = c) \cdot p(\Delta N_{tn} = c) \]

\[ = e^{-\lambda \Delta t} \sum_{c=0}^\infty \Phi \left( \frac{x_{tn} - \mu_{X_{tn} \mid X_{tn-1}}}{\sigma_{X_{tn} \mid X_{tn-1}}} \right) \frac{(\lambda \Delta t)^c}{c!}, \quad (3.6) \]

where \( \Delta t = t_n - t_{n-1}, \Delta N_{tn} = N_{tn} - N_{tn-1}, \phi(\cdot) \) is the standard normal density, \( \mu_{X_{tn} \mid X_{tn-1}} = x_{tn} + \beta \Delta t + c \mu_{\lambda}, \sigma_{X_{tn} \mid X_{tn-1}} = \Delta t + c \sigma_{\lambda}. \)

**Proposition 3.3.** The probability of no default at \( t_1 \), and density of \( X_{t_1} \) are given by

\[ P(\tau > t_1) = \sum_{c=0}^\infty P(X_{t_1} > 0 \mid \Delta N_{t_1} = c) \cdot p(\Delta N_{t_1} = c) \]

\[ = e^{-\lambda \Delta t} \sum_{c=0}^\infty \Phi \left( \frac{\mu_{X_{t_1}}}{\sigma_{X_{t_1}}} \right) \frac{(\lambda \Delta t)^c}{c!}, \quad (3.7) \]

\[ p(X_{t_1} = x_{t_1}) = \sum_{c=0}^\infty p(X_{t_1} > 0 \mid \Delta N_{t_1} = c) \cdot p(\Delta N_{t_1} = c) \]

\[ = e^{-\lambda \Delta t} \sum_{c=0}^\infty \phi \left( \frac{x_{t_1} - \mu_{X_{t_1}}}{\sigma_{X_{t_1}}} \right) \frac{(\lambda \Delta t)^c}{c!}, \quad (3.8) \]

where \( \Phi(\cdot) \) is the cumulative standard normal distribution, \( \phi(\cdot) \) is standard normal density, \( \mu_{X_{t_1}} = x_0 + \beta t_1 + k \mu_{\lambda}, \sigma_{X_{t_1}}^2 = t_1 + k \sigma_{\lambda}^2 \).

**Remark 3.1.** The recursion, given in above Propositions, is based on straightforward numerical integration, as the integrands are sums of standard normal densities. What is more, the sums converge quickly, since \( \frac{(\lambda \Delta t)^c}{c!} \to 0 \) rapidly as \( c \to \infty \), especially for \( \lambda \ll 1 \), as is the case of the calibration results given later in the paper, and for \( \Delta t = 0.25 \), i.e. for quarterly monitored defaults.

**Remark 3.2.** A similar recursion is suggested in [9]. The difference is that the recursion given by Fang et al. is backwards, i.e. first \( p(X_{t_{m-1}} = x_{t_{m-1}}, X_{t_m} > 0) \) is obtained, then \( p(X_{t_{m-2}} = x_{t_{m-2}}, X_{t_{m-1}} > 0, X_{t_m} > 0) \) and finally \( p(X_{t_1} = x_{t_1}, X_{t_2} > 0, \ldots, X_{t_{m-1}} > 0, X_{t_m} > 0) \). In our case, we first calculate \( p(X_{t_1} = x_{t_1}) \), then \( p(X_{t_2} = x_{t_2}, X_{t_1} > 0) \) and, in the end, \( p(X_{t_m} = x_{t_m}, X_{t_{m-1}} > 0, \ldots, X_{t_1} > 0) \), which gives straight away \( P(\tau > t_k) = P(X_{t_k} > 0, \ldots, X_{t_1} > 0) \) for each \( t_k \). The latter approach is more efficient for pricing CDSs, since for all monitoring times, \( t_k, k \geq 1 \), the survival probability is needed.
Figure 1: Survival probability over 5 years in the log-normal jump-diffusion model with $m$ default monitoring dates per year, where $m = 2, \ldots, 252$. Parameters are taken from a calibration to market data from 5 December 2008 (see Section 6 for details).

3.4 Continuous vs Discrete Default Monitoring

In some applications of structural default models, continuous monitoring of default is assumed, e.g. in [2], [15], in others, discrete monitoring, e.g. in [16], [5], [9]. In practice, a default is announced on a daily basis, as Fang et al. [9] argue.

In order to make models computationally more tractable, some studies, e.g. [16] and [5], assume that default is detected only on spread payment dates. We follow this line and thus assume that defaults are monitored quarterly.

In order to check the impact of this approximation, we calculate survival probabilities for $m = 2, \ldots, 252$ monitoring dates per year, i.e. from half-yearly to daily monitoring, at time $t = 5$, i.e. at the maturity of CDSs that we will use later to price CDOs. The results are shown in Fig. 1. As expected, the survival probabilities converge and are a decreasing function of $m$. The difference between the survival probability calculated for $m = 4$, i.e. for the number of monitoring times we will assume in the computations, compared to $m = 252$, i.e. the maximum realistic monitoring times per year, is 0.59 percentage points.

Remark 3.3. The question of continuous versus discrete monitoring of defaults is mathematically equivalent to that of continuous versus discrete barrier monitoring in pricing barrier options. Broadie, Glasserman and Kou [4], in a diffusion model, approximate prices of discrete barrier options by formulas for continuous barrier options but with a shifted barrier, precisely (Theorem 1.1 in [4])

$$V_m(H) = V\left(H e^{-\beta \sigma \sqrt{\Delta T}}\right) + o\left(\frac{1}{\sqrt{m}}\right),$$

where $V_m(H)$ is the price of discretely monitored knock-out call option, $V(H)$ is the price of a corresponding continuously monitored barrier option, $\beta$ is a constant derived by the
authors, \( m \) is the number of monitoring times at intervals \( \Delta T \). This correction accounts for the probability of barrier crossings between monitoring dates.

In a finite activity jump model, as the probability of a jump in an interval of length \( \Delta T \) is \( O(\Delta T) \), and that of an undetected down-up combination even \( O(\Delta T^2) \), we conjecture similar asymptotic behaviour.

As we calibrate the initial distance-to-default \( X_0 \) to CDS spreads, which are functions of the survival probabilities, this correction is implicitly invoked, as seen from (3.3), when we use the calibrated model later to price basket derivatives based on the same monitoring dates. This makes the difference between monitoring frequencies a higher order effect.

4 A Multi-Name Structural Model

In this section, we extend the single-name jump-diffusion to a multi-name model, with the view of pricing basket credit derivatives.

The works closest to our basic setup are [32] and [19], where [32] uses a log-normal jump-diffusion multi-name model to price CDOs, while [19] use a double exponential jump distribution.

4.1 The Model Setup

Consider a portfolio consisting of CDSs on \( N \) different companies. Under the risk neutral measure, asset values are driven by a log-normal jump diffusion

\[
\frac{dA_i^t}{A_i^t} = (r - \lambda \nu) \, dt + \sqrt{1 - \rho^2} \, \sigma dW_i^t + \sqrt{\rho} \, \sigma dM_t + (Y - 1) \, dN_i^t, \tag{4.1}
\]

where \( M_t \) is a standard Brownian motion, \( \rho \in [0, 1) \), \( \ln Y \sim N(\mu_Y, \sigma_Y^2) \), and \( W^i \) as before and mutually independent with \( M_t \), \( N_t \). Like in the univariate case, we consider the distance-to-default \( X_i^t = \frac{1}{\sigma} (\ln(A_i^t) - \ln(B_i)) \), where \( B_i \) is a constant default barrier. The dynamics of \( X_i^t \) are given by

\[
dX_i^t = \beta \, dt + \sqrt{1 - \rho^2} \, \sigma dW_i^t + \sqrt{\rho} \, \sigma dM_t + \Pi \, dN_i^t, \tag{4.2}
\]

where \( \beta = (r - \lambda \nu - \frac{1}{2} \sigma^2)/\sigma \), \( \Pi \sim N(\mu_{\Pi}, \sigma_{\Pi}^2) \), \( \mu_{\Pi} = \mu_Y/\sigma \), \( \sigma_{\Pi}^2 = \sigma_Y^2/\sigma^2 \). \( W^i \) is an idiosyncratic factor, specific to each company, which affects the value of each company’s assets alone, such as the management of a company. \( M, N \) and \( \Pi \) represent global factors that affect the default environment of all companies. As Hull et al. [16] note, such a global factor could be for instance the S&P 500 index. Other candidates could be GDP, or more specifically investment spending, central bank interest rates or the unemployment rate, since they show in which phase of the business cycle the economy is.

The jumps \( N \) together with \( \Pi \), model sudden effects by global factors on the situation of companies, while \( M \) models more gradual influences. We aim at taking into account predominantly negative jumps and consider only \( \mu_{\Pi} < 0 \). The intensity of \( N \), \( \lambda \), then measures the frequency of economic shocks. If the absolute value of the jump mean is high, then \( \lambda \) can be interpreted as the frequency of economic crises. A similar interpretation can be found in [32].

Finally, the distribution of distances-to-default \( X_{i0} \) indicates the current state of the market. If the mass of the distribution is close to zero, then the financial situation of
companies is relatively bad, if the mass is concentrated far away from zero, it shows that the economic climate is good.

It is worth discussing in more detail a main premise of the model, namely that the firms are assumed exchangeable, specifically that the volatility parameter \( \sigma \) is identical for all firms. Credit indices usually consist of companies from different industry sectors and different regions, and clearly there will be differences between the variances of asset values. If we had started with individual \( \sigma^i \) for each company in the portfolio, the dynamics would be, instead of (4.1),

\[
\frac{dA^i_t}{A^i_t} = (r - \lambda \nu) \, dt + \sqrt{1 - \rho} \, \sigma^i \, dW^i_t + \sqrt{\rho} \, \sigma^i \, dM_t + (e^{\sigma^i \Pi} - 1) \, dN_t, \tag{4.3}
\]

where \( \sigma^\Pi \) and \( \mu^\Pi \) still have the same values for all companies. In terms of \( X^i_t \),

\[
dx^i_t = \beta^i \, dt + \sqrt{1 - \rho} \, dW^i_t + \sqrt{\rho} \, dM_t + \Pi \, dN_t, \tag{4.4}
\]

where \( \beta^i = (r - \lambda \nu - \frac{1}{2} \sigma^2)/\sigma^i \), and the initial distance-to-default is \( X^i_0 = (\ln(A^i_0) - \ln(B^i_0))/\sigma^i \). The volatility dependence of the default probabilities, is therefore to a large extent captured by the initial credit quality through \( X^i_0 \), and, especially for not too long maturities, the effect of the drift will be negligible compared to the diffusive and jump components. The initial \( X^i_0 \) will be calibrated to individual CDS spreads allowing through this heterogeneity in the portfolio.

We will discuss the role of correlation in the next section.

### 4.2 Dependence Structure of Companies in the Model

The dependence between companies in the model is determined by diffusion correlation, represented by a single parameter \( \rho \), together with perfect jump correlation for all companies, represented by common jump times of \( N_t \) and the same jump sizes, \( \Pi \). One might consider to allow different exposure to market factors by individual companies, which we do not do here for ease of calibration and computation, and since we are more concerned with the macroscopic behaviour of the basket.

In the present setting, the dependence between companies can vary from slight, if the values of \( \rho, \lambda, |\mu^\Pi|, \sigma^\Pi \) are small, to extremely strong, if the values are high. A few years ago, taking into account such a strong dependence might have been regarded as unrealistic, but the recent credit crunch, which affected not only financial markets but also the global real economy, showed that the dependence between companies can indeed be very strong.

In order to measure this dependence, we calculate an overall correlation as in [19] and [23], indicating common movements of companies’ distance-to-default, and discuss default correlation, measuring common defaults of firms.

**Proposition 4.1.** The overall correlation between \( X^i_t \) and \( X^j_t \), driven by (4.2), is

\[
\rho_{X^i_tX^j_t} = \frac{\rho + \zeta}{1 + \zeta}, \tag{4.5}
\]

where \( \zeta = \lambda(\sigma^2 + \mu^2) \).
Remark 4.1. Since $\rho \in [0, 1)$, $\lambda > 0$, $\rho_{X_i^tX_j^t} \in [0, 1)$. The correlation increases both with diffusion correlation $\rho$ and with the jump parameters $\lambda$, $\mu_\Pi^2$, $\sigma_\Pi^2$. $\rho_{X_i^tX_j^t} = 0$ when $\rho = 0$ and $\zeta = 0$ (i.e. $\lambda = 0$ or $\sigma_\Pi^2 + \mu_\Pi^2 = 0$), while $\rho_{X_i^tX_j^t} \to 1$ when $\rho \to 1$ or $\zeta \to \infty$ (i.e. $\lambda \to \infty$, $\sigma_\Pi^2 \to \infty$, or $\mu_\Pi^2 \to \infty$).

The crucial indicator that shows the tendency of companies to default together is default correlation, defined in [34] as

$$\text{Corr}[D_1(t), D_2(t)] = \frac{E[D_1(t) \cdot D_2(t)] - E[D_1(t)] \cdot E[D_2(t)]}{\sqrt{\text{Var}[D_1(t)]} \cdot \sqrt{\text{Var}[D_2(t)]}},$$

where $D_i(t)$ is given by

$$D_i(t) = \begin{cases} 1 & \text{if firm } i \text{ defaults by time } t, \\ 0 & \text{otherwise.} \end{cases}$$

For a diffusion model, [34] obtains an analytical formula to evaluate $\text{Corr}[D_1(t), D_2(t)]$, while for a jump-diffusion model default correlation can be calculated only numerically. In order to evaluate it accurately, a very high number of simulations has to be used, especially for small $t$, since in that case the probability of a joint default of two companies is extremely small.

Lipton and Sepp [23] argue that for jump-diffusion models, the overall correlation is closely related to default correlation. As a more differentiated indicator of default correlation we propose a crisis scenario correlation, defined in the following Proposition.

**Proposition 4.2.** We define crisis scenario correlation as correlation between distances to default, $X_i^t$ and $X_j^t$, driven by (4.2), given that in $(0, s)$, $s \leq t$, $k$ negative jumps occurred. It is given by

$$\rho_{X_i^tX_j^t}^{CS} = \rho + \kappa, \quad (4.6)$$

where $\kappa = \tilde{k} \text{Var}[\Pi | \Pi < 0] + \Delta \tilde{t} \zeta$, $\tilde{k} = k/t$, $\Delta \tilde{t} = (t - s)/t$, $\zeta = \lambda(\sigma_\Pi^2 + \mu_\Pi^2)$.

The coefficient shows the dependence between companies in the portfolio when market shocks already have happened within the lifetime of a contract. It therefore gives an indication of how high default correlation might be.

### 4.3 The Large Basket Limit

Willemann [32] derives analytical approximations to the survival probability for each company in the basket, and semi-analytical approximations to the portfolio loss distribution. In contrast, Kiesel and Scherer [19] simulate the asset value process for each company to estimate joint default probabilities, which is not subject to model simplifications, but computationally expensive, since the basket size is typically large, e.g. 125 for the iTraxx index. We make a virtue of this by using a large basket approximation in the spirit of [5], extending the model to jump-diffusions. Using this approach, the computational effort is independent of the number of companies. We will test the validity of the large basket approximation in 4.4, and its numerical solution in Section 5.
Following [5], we study the evolution of the empirical measure of \( \{ X_i^t, i = 1, \ldots, N \} \),
\[
\nu_{N,t} = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i^t},
\]
where \( \delta_x \) is a Dirac measure centred at \( x \). We are particularly interested in the limiting behaviour of \( \nu_{N,t} \) for large \( N \),
\[
\nu_t = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i^t}.
\]

In the diffusion case, \( \lambda = 0 \) in 4.2, [22] show that for exchangeable \( X_0^i \), under further mild conditions on the distribution of \( X_0^i \), the limit \( \nu_t \) exists and its density is the unique solution of the stochastic partial differential equation
\[
dv(t, x) = -\beta v_x(t, x) \, dt + \frac{1}{2} v_{xx}(t, x) \, dt - \sqrt{\rho} v_x(t, x) \, dM_t.
\]
It is a straightforward application of Itô’s lemma [5] to see that a solution is given by
\[
v(t, x) = u(t, x - \sqrt{\rho} M_t),
\]
if \( u(x, t) \) satisfies the deterministic PDE
\[
u_t = \frac{1}{2} (1 - \rho) u_{xx} - \beta u_x
\]
with initial condition \( u(0, x) = v_0 \).

The intuitive explanation is that in the large basket limit, the independent drivers have averaged to a deterministic diffusion, while the market factor \( M \) drives the whole basket.

Adding finite activity jumps, as per (4.2), we get the following result.

**Proposition 4.3.** The limit empirical measure \( \nu_t \) defined in (4.8) (exists and) and has a density \( v \) of the form
\[
v(t, x) = u(t, x - \sqrt{\rho} M_t - J_t),
\]
where \( J_t = \sum_{i=0}^{N_t} \Pi_i \), and \( u \) solves (4.10). Moreover, \( v \) satisfies the SPDE
\[
dv(t, x) = -\beta v_x(t, x) \, dt + \frac{1}{2} v_{xx}(t, x) \, dt - \sqrt{\rho} v_x(t, x) \, dM_t
\]
\[
+ v(t, x - \Pi) \, dN_t - v(t, x) \, dN_t.
\]

In the present setting, we are especially interested in losses, i.e. barrier crossings of \( X_i^t \). Bush at al. [5] derive an initial boundary value problem for the purely diffusive SPDE for the case of a continuously active default barrier; [17] gives an extension to the jump case.

For computational convenience, we make the assumption that defaults are only observed at a discrete set of times, which we take quarterly to coincide with the payment
dates. If a firm’s value is below the default barrier on one of the observation dates \(T_i\), it is considered defaulted and removed from the basket, so we use (4.2) in \([0, \tau_i]\), and
\[
X_t^i = -\infty \quad t > \tau_i^i, \\
\tau_i^i = \inf\{t \in \{T_1, \ldots, T_n\} : X_t^i \leq 0\}. \tag{4.13}
\]

We therefore solve the following SPDE problem:
\[
dv = -\beta v_x dt + \frac{1}{2} v_{xx} dt - \sqrt{\rho} v_x dM_t \\
+ v(t_-, x - \Pi) dN_t - v(t_-, x) dN_t \quad t \in (T_k, T_{k+1}), 0 \leq k < n \tag{4.14}
\]
\[
v(0, x) = v_0(x) \\
v(T_k, x) = 0 \quad \forall x < 0, 0 < k \leq n \tag{4.15}
\]

We discuss the numerical implementation in Section 5.

4.4 Validity and Benefits of the Large Basket Approximation

In this section, we analyse if the size of the CDO basket, which is typically \(N = 125\), is large enough to use the limiting SPDE (4.12) obtained for \(N \to \infty\). We construct “nested” baskets of size \(N_k = 5^k\), \(k = 1, \ldots, 9\), i.e. the \(N_k\) basket contains all firms of the \(N_{k-1}\) basket etc. For each basket, we calculate expected tranche losses, \(E[Y_t]\), where
\[
Y_t = [L_t - a]^+ - [L_t - d]^+,
\]
and the losses from either (2.1) by direct Monte Carlo simulation of (4.13), (4.2), or by
\[
L_t = \int_{-\infty}^{\infty} v(t, x) dx, \tag{4.17}
\]
where \(v\) is the solution to the SPDE.

In either case, we need to simulate initial distances-to-default \(X_0^n\), which either serve directly as starting values for (4.2), or constitute the initial distribution \(\nu_{N,0}\) for the SPDE as in (4.7). Let \(t = 0\) denote the starting date for the CDO. We assume that at time \(t = -m\), the distances-to-default, \(X_{-m}^n\), were independent and normally distributed with mean \(\mu_{X_{-m}}\) and standard deviation \(\sigma_{X_{-m}}\), and that for \(t > -m\), \(X_t^n\) follows a jump-diffusion process given by (4.2), with model parameters taken from calibration results for 5th December 2008 (see Section 6 for details). Then, at time \(t = 0\),
\[
X_0^n = X_{-m}^n + \Delta G_0 + \Delta Z_0^n, \tag{4.18}
\]
where \(\Delta G_0\) is the increment of global factors over time \((-m, 0]\), \(\Delta Z_0^n\) is the increment of individual factors. We simulate \(X_0^n\) in two stages, i.e. we first simulate a realisation of common factors, and then \(X_{-m}^n\) together with \(\Delta Z_0^n\), i.e. conditional on \(\Delta G_0 = g\),
\[
X_0^n \sim N\left(\mu_{X_{-m}} + \beta m + g, \sigma_{X_{-m}}^2 + (1 - \rho) m\right). \tag{4.19}
\]

For Monte Carlo simulation of the basket, one needs to simulate \(N_{\text{sims}} \times N \times n\) realisations of individual factors, plus \(N_{\text{sims}} \times n\) realisations of global factors, where \(n\) is the number of monitoring dates and \(N_{\text{sims}}\) the number of Monte Carlo runs.
Calculating the tranche losses accurately by direct Monte Carlo simulation is computationally costly for a CDO on a basket of size $N = 125$, and would be infeasible for our numerical studies with $N$ as big as 2 millions. To deal with this, we developed a heuristic algorithm based on Multi-level Monte Carlo simulation [11], where we use smaller baskets as control variates for large baskets. The details are outside the scope of this article and the subject of current research.

For the SPDE, only global factors need to be simulated, making the complexity of the method independent of the basket size. We solve the SPDE numerically via a combination of finite differences and Monte Carlo simulation as detailed in Section 5.

In summary, the difference between results obtained by the two methods is due to: error of the finite difference discretisation of the SPDE; simulation error, both for the SPDE and direct Monte Carlo; large basket approximation. We want to focus on the latter effect, therefore we choose a high number of grid points to reduce the discretisation bias, and a high number of simulations to reduce the Monte Carlo error to a negligible size.

As seen in Fig. 2, the difference between the SPDE results and those from direct Monte Carlo simulation of the basket goes to zero for large $N$.

The SPDE results depend on $N$ only via the initial condition. Therefore, the results in Fig. 2 indicate that the approximation error of tranche spreads due to the limit dynamics, i.e. the difference between the SDE and SPDE solution, is noticeably smaller than the effect of the finite sample for the starting values, i.e. the difference between the SPDE solution for finite $N$ and its asymptotic limit, which corresponds to a continuous initial density.

Closer inspection reveals that this is due to the fact that the tails of the initial distribution of $X_{n0}$ are not well resolved for moderate $N$, which affects junior tranches, which measure losses in the left tail, more than senior tranches. To verify that numerically, we repeated the experiment with all mass centred in a single point $X_{n0}^n = z$, where $z$ is a constant, for all firms in the portfolio. Thus the initial condition becomes independent of the basket size. This gave a significantly smaller difference, especially for junior tranches.

For $N = 125$, i.e. $k = 3$, the results of both methods are close, hence we argue that this number of companies is sufficiently large to use the large basket approximation.

5 Numerical Implementation

The key ingredient in the calculation of index and tranche spreads are the basket losses, which can be obtained from the large basket density $v$ as (4.17), where $v$ has to be approximated by a numerical solution to the SPDE. In this section, we outline a numerical method based on Monte Carlo simulation on top of a PDE solver. In the presence of non-smooth initial data, and with senior tranche prices depending on very low probability events, a carefully adapted numerical scheme is necessary.

We discuss the calibration of model parameters, including initial conditions, in Section 6, and assume these as given here.

5.1 Setup and Finite Difference Discretisation

We can take advantage of the representation (4.11) of the solution to the SPDE problem (4.14) to (4.16) piecewise in time, as the boundary condition is not active in intervals
Figure 2: Expected losses in tranches $[0\%-3\%]$, $[3\%-6\%]$, $[6\%-9\%]$, $[9\%-12\%]$, $[12\%-22\%]$, $[22\%-100\%]$, calculated by basic Monte Carlo simulation of $N_k = 5^k$ asset value processes, $k = 1, \ldots, 9$, and the SPDE approximation.
\((T_k, T_{k+1})\), and therefore the Brownian driver and the jump part only introduce shift to the solution, accumulated over the entire interval. Conditional on the number of jumps occurred in the interval, the shift is normally distributed. Using Proposition 4.3, the problem can therefore be written as

\[
v(t, x) = \begin{cases} 
  v^{(k)}(t - T_k, x - \sqrt{\rho}(M(t) - M(T_k)) - (J(t) - J(T_k))), & 0 \leq 0, t = T_{k+1}, \\
  0, & \text{else if } t \in (T_k, T_{k+1}].
\end{cases}
\]

(5.1)

for \(0 \leq k < n\), where \(J_t = \sum_{i=0}^{N(t)} \Pi_i\), \(v^{(k)}\) is the solution to the (deterministic) problem

\[
v_t^{(k)} = \frac{1}{2}(1 - \rho)v_{xx}^{(k)} - \beta v_x^{(k)}, \quad t \in (0, \Delta T) = (0, T_{k+1} - T_k)
\]

(5.2)

\[
v^{(k)}(0, x) = v(T_k, x)
\]

(5.3)

assuming payment dates are equally spaced with intervals \(\Delta T = T_{k+1} - T_k\).

This suggests the following inductive strategy for \(k = 0, \ldots, n - 1\):

1. Start with \(v^{(0)}(0, x) = v_0(x)\).
2. Solve the PDE (5.2) numerically in the interval \((0, T_1)\), to obtain \(v^{(0)}(T_1, x)\).
3. Simulate \(M(T_1), J(T_1)\) evaluate \(v(T_1, x)\) according to (5.1).
4. For \(k > 0\), having computed \(v(T_k, x)\) in the previous step, use this as initial condition for \(v^{(k)}\), and repeat until \(k = n\).

To solve the PDEs (5.2) by a finite difference method, we approximate the distribution by one with support \([x_{\text{min}}, x_{\text{max}}]\), and set asymptotic boundary conditions \(v(t, x_{\text{min}}) = v(t, x_{\text{max}}) = 0\). Since the initial distribution is assumed localised (as it approximates the distribution of a finite number of firms), the localisation error for a given path \(M\) can be made as small as needed by a suitable choice of \(x_{\text{min}} < 0\) and \(x_{\text{max}} > 0\). We can moreover ensure that the expected error of this approximation is much smaller than the standard error of the Monte Carlo estimates.

Then, introduce a grid \(x_0 = x_{\text{min}}, x_1 = x_{\text{min}} + \Delta x, \ldots, x_n = x_{\text{min}} + J\Delta x\), where \(\Delta x = (x_{\text{max}} - x_{\text{min}})/I\), timesteps \(t_0 = 0, t_1 = \Delta t, \ldots, t_I = I\Delta t\), where \(\Delta t = \Delta T/I\), and define an approximation \(v^j_i\) to \(v(t_i, x_j)\) by the difference scheme

\[
\frac{v^j_i - v^{j-1}_i}{\Delta t} = \theta \left\{ \frac{1}{2} \frac{v^j_{i+1} - 2v^j_i + v^{j-1}_i}{\Delta x^2} - \beta \frac{v^j_{i+1} - v^{j-1}_i}{2\Delta x} \right\} + (1 - \theta) \left\{ \frac{1}{2} \frac{v^{j-1}_{i+1} - 2v^{j-1}_i + v^{j-1}_j}{\Delta x^2} - \beta \frac{v^{j-1}_{i+1} - v^{j-1}_j}{2\Delta x} \right\}.
\]

(5.4)

(5.5)

For an introduction to finite differences in financial instrument pricing see e.g. [31]. The scheme is of second order accurate in \(\Delta x\). The backward Euler scheme \(\theta = 1\) is of first order accurate in \(\Delta t\) and strongly A-stable. The Crank-Nicolson scheme \(\theta = \frac{1}{2}\) is of second order accurate, and is unconditionally stable in the \(l_2\)-norm for sufficiently smooth solutions, but gives rise to instabilities for initial conditions comprising \(\delta\)-distributions like in the present setting. We address this, together with aspects of approximating non-smooth initial and interface conditions (4.16) accurately on the grid, in the following section.
5.2 Initial and Interface Conditions

The initial condition has the form

\[ v(0, x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - x^i), \]

where \( x^i \) is the initial distance-to-default of firm \( i \). A simple approach to approximate this distribution on the grid could bin the firms into symmetric intervals of width \( \Delta x \) around grid points, and assign the fraction of firms in such an interval to the corresponding grid value, that is

\[ v_0^j = \frac{1}{N \Delta x} \int_{x^j - \Delta x/2}^{x^j + \Delta x/2} \delta(x^i - x) \, dx \]

where the factor \( \Delta x^{-1} \) ensures the right scaling of the density. This approximation cannot distinguish between initial positions \( x^i \) in an interval of length \( \Delta x \), which reduces the overall order of the finite difference scheme to one.

To achieve higher, i.e., second order accuracy, the \( \delta \)'s have to be “split” between adjacent grid points. The correct weighting for a single firm with distance-to-default \( x^i \in [x^j, x^{j+1}) \), is

\[ v_0^k = \begin{cases} \Delta x^{-2}(x^j+1 - x^i), & k = j, \\ \Delta x^{-2}(x^i - x^j), & k = j + 1, \\ 0, & \text{else}. \end{cases} \]

The extension to more firms is obvious, and can be written more elegantly as an \( L_2 \)-projection of the initial condition onto the basis of linear splines \( \langle \Phi_k \rangle_{0 \leq k \leq N}, \)

\[ v_0^k = \langle \Phi_k, v_0 \rangle = \int_{x_{\min}}^{x_{\max}} \Phi_k(x)v_0(x) \, dx \]

where

\[ \Phi_k(x) = \frac{1}{\Delta x} \min \left( \max(x - x_k + \Delta x), \max(x - x_k - \Delta x) \right). \]

By this construction, \( \Delta x \sum_{k=0}^{N} v_0^k = 1 \). See also [26] for applications of this idea to option pricing and estimation of sensitivities.

To incorporate the interface condition (5.1) at \( t = T_k \), one has to evaluate the grid function at shifted arguments \( x_j - \sqrt{\rho}(M(t) - M(T_k)) - (J(t) - J(T_k)) \), which do not normally coincide with grid points. To deal with this, we first define a piecewise linear interpolant from the approximation \( \hat{v} \), obtained in the last step over the previous interval \( [T_{k-1}, T_k], v_j^{(k),I}, I \) the number of timesteps, by

\[ \hat{v}(T_k, x) = \sum_{j=0}^{N} \Phi_j(x - \Delta M - \Delta J)v_j^{(k),I}, \quad (5.6) \]

where \( \Delta M = M(T_k) - M(T_{k-1}), \Delta J = J(T_k) - J(T_{k-1}) \). Then, approximate (5.1) by setting

\[ v_j^{(k+1),0} = \int_{\max(x_j - \Delta x/2,0)}^{x_j + \Delta x/2} \hat{v}(T_k, x) \, dx, \quad (5.7) \]
and use this as initial condition for the next interval. This ensures that
\[
\Delta x \sum_{j=0}^{J} \psi_j^{(k+1),0} = \int_0^{x_{max}} \hat{v}(T_k, x) \, dx,
\]
so the cumulative density of firms with firm values greater than 0 is preserved. It also
has the effect that the solution is smoothened at \( x = 0 \). In contrast to simpler, e.g.
pointwise application of the interface condition, this procedure guarantees second order
convergence in \( \Delta x \). See [26] for averaging procedures to restore higher order convergence
for non-smooth payoffs in option pricing.

Finally, it is well-known that Crank-Nicolson timestepping has reduced convergence
order for discontinuous initial conditions, and does not converge at all for Dirac initial
conditions, unless the timesteps are chosen very small. This severely slows down the
performance.

A simple and well-established remedy is to replace the first Crank-Nicolson steps
with backward Euler steps, a practice now known as Rannacher timestepping [28]. The
analysis in [12] shows that the optimal balance between accuracy and stability is achieved
by replacing the first two Crank-Nicolson steps by four backward Euler steps of half the
stepsize. We do this at \( t = 0 \), and also at \( t = T_k \) where the interface conditions introduce
discontinuities at \( x = 0 \).

### 5.3 Loss Simulation

For a given realisation of the market factors, we can approximate the loss functional (5.1)
at time \( T_k \), using (5.6) and (5.7), by
\[
\hat{L}_{T_k} = 1 - \int_0^{x_{max}} \hat{v}(T_k, x) \, dx = 1 - \Delta x \sum_{j=1}^{J-1} \psi_j^{(k+1),0}.
\]
(5.8)

We first consider the discretisation error in \( \Delta t \) and \( \Delta x \), by fixing the path in (5.8).
Fig. 3 shows, for a typical set of model parameters, how many grid points (\( I \)) and how

![Figure 3: Extrapolation-based estimator for the discretisation error of \( \hat{L}_{T_n} \) for increasing \( J \) (left) and \( I \) (right) for a single realisation of the path \( M \).](image)

many timesteps between payment dates (\( J \)) are necessary for a desired accuracy \( \epsilon \). We
clearly see second order convergence in $\Delta t$ and $\Delta x$. Note that the time smoothing scheme, projection of initial positions, and adapted averaging, are essential to achieve this.

Next, we want to compute expected losses and tranche losses. If we explicitly include the dependence on the Monte Carlo samples $\Phi = (\Phi^i_1)_{1 \leq i \leq n}$ in $\hat{L}_{T_k}(\Phi)$, where $\Phi^i$ are drawn independently from a standard normal distribution, then for $N_{\text{sims}}$ simulations with samples $\Phi^i_k = (\Phi^i_{k})_{1 \leq k \leq n}$,

$$
E^Q[X^T_k] \approx E^Q[\max(d - \hat{L}_{T_k}, 0) - \max(a - \hat{L}_{T_k}, 0)]
$$

$$
\approx \frac{1}{N_{\text{sims}}} \sum_{l=1}^{N_{\text{sims}}} \left( \max(d - \hat{L}_{T_k}(\Phi^i), 0) - \max(a - \hat{L}_{T_k}(\Phi^i), 0) \right).
$$

We expect (weak and strong) convergence for this estimator of order $\Delta x^2 + \Delta t^2$, which is confirmed by numerical tests. In the following we fix $\Delta x$ and $\Delta t$ at values which have proven empirically to give negligible discretisation error.

For the following simulations we have used two data sets: from 22 February 2007 and 5 December 2008. Initial positions for individual firms were calibrated to their individual CDS spreads, and were well within the range $[x_{\text{min}}, x_{\text{max}}] = [-10, 20]$.

We now analyse the convergence of the obtained Monte Carlo estimator. The variance of estimated expected tranche losses is given in Figure 4. For senior tranches, the variance is higher than for equity and mezzanine ones, especially for 22 February 2007. The reason behind this is that a large number of companies have to default in order to affect senior tranches and such an event is rare. For 22 Feb 2007, the estimated parameters imply normal market conditions, where multiple defaults are highly unlikely (see Section 6 for a discussion of calibration results). In order to price senior tranches more accurately, variance reduction techniques such as importance sampling should be applied, see e.g. [6], however the accuracy of the results for the basic scheme was found sufficient for the purposes of this study.

6 Calibration and Results

The first step in using a model is to estimate its parameters by calibrating the model to market data. We fit the jump-diffusion model and also, for comparison, the diffusion model, to CDO index and tranche spreads.

6.1 Stating the Calibration Problem

Our calibration exercise for model parameters from 4.1 can be stated as follows.

**Problem 1.** Given market spreads at time $t = 0$, of CDO tranches, $C^0_j(T_i)$, and CDO index, $C_0(T_i)$, for maturities $T_i$, $i = 1, \ldots, M$, tranches $j = 1, \ldots, G$, and given spreads $c_0 = (c^1_0, \ldots, c^N_0)$ for CDSs written on underlying companies $n = 1, \ldots, N$, solve the minimisation problem

$$
\sum_{i=1}^{M} \sum_{j=1}^{G} \alpha^j_i \left( C^0_i(T_i) - C^j_{\theta,x^j_0}(T_i) \right)^2 + \sum_{i=1}^{M} \alpha^i \left( C_0(T_i) - C_{\theta,x^0_0}(T_i) \right)^2 \rightarrow \min_{\theta}, \tag{6.1}
$$

where $\theta = (\rho, \sigma, \lambda, \sigma_{\Pi}, \mu_{\Pi})$, subject to
Figure 4: Monte Carlo estimators with 95%-confidence intervals for expected losses in tranches \([0\%-3\%], [3\%-6\%], [6\%-9\%], [9\%-12\%], [12\%-22\%], [22\%-100\%]\), for \(N = 4k+1\) simulations, \(k = 1, \ldots, 9\). The results are for parameters from a calibration of the model to data from 22 February 2007 (see Section 6).
(i) \( x_0 = (x_0^1, \ldots, x_0^N) \) is a solution to
\[
c_0 = c_0^\theta(x_0),
\]
(6.2)

(ii) \( \rho \in [0, 1), \sigma > 0, \lambda > 0, \sigma_{II} > 0, \mu_{II} < 0, \)
where \( c^\theta, C_0^{\theta,x_0}, C_1^{\theta,x_0} \) denote CDS, CDO tranche and index spreads calculated in the model under the risk neutral measure, \( x_0 \) is a vector of initial distances-to-default, \( \alpha = (\alpha_i^j, \alpha_i^0) \) is a scaling vector.

Our calibration exercise is a minimisation problem with scaled least-squares objective function, linear inequality constraints (ii) and non-linear equality constraints (i).

The number of calibration prices, \( M \cdot (G + 1) \), is typically much larger than the number of parameters, here 5. Since CDO tranche and index spreads have different orders of magnitudes, we scale the data by \( \alpha \) to make each observation equally important. Alternatively, one could fit the parameters to match the most liquid products exactly, or give those more weight in the objective function.

The non-linear constraints are incorporated directly into (6.1) by numerical inversion of (6.2) giving \( x_0 = x_0(\sigma, \lambda, \sigma_{II}, \mu_{II}) \). Given a vector of initial distances-to-default, we solve the minimisation problem (6.1) with bounds on the parameters. A robust algorithm developed specially for this kind of minimisation problems is the interior-reflective Newton method by Coleman and Li (see [7], [8] for details). We use this algorithm as implemented in the Matlab Optimisation Toolbox.

### 6.2 Computational Issues

For each calibration, a number \( \gamma \) of iterations is needed to solve (6.1), and in each iteration, an approximation to the objective function and its derivative is required.

Since we use finite differences to approximate the derivatives, \( x_0^\gamma \) is searched \( K = (1 + 2b) \times N \times \gamma \) times, where \( N = 125 \) is the number of CDSs in the portfolio, \( b = 4 \) is the number of parameters in the model except \( \rho \) (\( \rho \) does not occur in the calculations of \( x_0 \)), and \( \gamma \) is in the range of \( 10 - 20 \). Since \( K \approx 1.5 \times 10^4 \), an efficient way of finding \( x_0 \) is crucial.

What is more, the objective function (6.1) is estimated by a Monte Carlo method, and in order to obtain accurate results, a high number of simulations is needed, as seen from the accuracy of expected tranche losses in Fig. 4. It seems natural to balance the Monte Carlo variance against the accuracy the current iterate.

We address these issues in the following.

#### 6.2.1 Search for Initial Distances-To-Default

Given CDS data, \( c_0, \) which is an \( N \) vector, we search for an \( N \) vector of initial distances-to-default, \( x_0 \). To enhance the speed of the calculations, we take advantage of properties of \( c_0 \). CDS spreads decrease with the initial distance-to-default, since higher \( x_0 \) leads to higher survival probability, which entails lower spreads, i.e. if \( c_0^i > c_0^j \) then \( x_0^i < x_0^j \). Hence, for a CDS spread, \( c_0^s \), where \( c_0^{\min} \leq c_0^s - 1 < c_0^s < c_0^{s+1} \leq c_0^{\max} \), the initial distance-to-default, \( x_0^s \), is in \( (x_0^{s+1}, x_0^{s-1}) \). Hence, a good starting point for an algorithm searching
for $x_0^s$ is either $x_0^{s+1}$ or $x_0^{s-1}$. Sorting spreads, then choosing starting points as stated above, greatly decreases calculation time of $x_0$.

6.2.2 Monte Carlo “Inside” the Objective Function

Our aim is to calibrate the model using a sufficiently high number of paths such that the simulation error is negligible. Given the numerical evaluation of the objective function and its derivatives is part of an iterative solution method for the optimisation problem, where the initial parameter iterates will be inaccurate, it is unnecessary to evaluate the spreads there to high accuracy.

Let $N_i, i = 1, \ldots, Z$, be an increasing sequence of numbers of simulations, and let $\theta_{N_i}$ be a sequence of parameters obtained using $N_i$ simulations, with an iterative scheme with starting value $\theta_{N_{i-1}}$ and $\gamma_i$ iterations. Since the Monte Carlo estimator converges, it is hoped that the $\theta_{N_i}$ converge to the optimiser, while only few iterations are needed for large $N_i$. This idea is a simple version of the Multi-layer method in [18].

In the calibration exercises, we heuristically picked $N_1 = 6 \times 10^3$, since for this number of simulations the estimator “starts converging”, as can be observed in Fig. 4, and the computation time per iteration is very low. For $N_1$, $\gamma_1$ is about 15, whereas for $N_3 = N_{Z-1} = 10^5$, it is about 2-4, and, finally, for $N_4 = 10^6$, it is only 1. The overall computation time of the calibration algorithm is much reduced compared to calibration with uniformly $N_Z$ simulations.

It would be possible to automatise this procedure and find an optimal sequence $N_i$ to minimize the overall computational time, based on the variance of the Monte Carlo estimator and the convergence speed of the optimisation algorithm.

6.3 Calibration Results

In order to check if the model is flexible enough to match market spreads both in quiet and extreme market conditions, we calibrate it to pre-crisis data, from 22 February 2007, and during the crisis, 5 December 2008. We do not use current data since the market is not sufficiently liquid.

For the first data set (see Table 1), spread curves both for index and tranches are increasing with time, super senior tranches (22%-100%) are close to zero, hence the market does not expect any shocks in the near future. For the second period (see Table 4), all quotes are much higher (for example prices of super senior tranches are higher than junior mezzanine (3%-6%) with maturity 5 years for 22 February 2007), the index spread curve is inverted, curves for tranches are almost flat, hence the market anticipates that the situation will get worse in the near future. We anticipate that parameters indicating market turbulences will be considerably higher for the second data set.

6.3.1 Measures of Fit

We compare the calibration results by the ARPE (Average Relative Percentage Error),

$$ARPE = \frac{1}{s} \sum_{i=1}^{s} \left| \frac{y_i - y_i^\theta}{y_i} \right|,$$

(6.3)
Table 1: Calibration results for 22 February 2007, iTraxx Main Series 6 index. Units are basis points (bps) if not stated otherwise. Estimated parameters are given in Table 2, measures of fit in Table 3. We assume that $r = 0.042$, $R = 0.4$.

and the RMSE (Root Mean Square Error),

$$RMSE = \sqrt{\frac{1}{s} \sum_{i=1}^{s} (y_i - y_i^\theta)^2}, \quad (6.4)$$

where $y$ is a vector of observed prices, $y_i^\theta$ is a vector of prices obtained from the model, $s$ the total number of calibration prices. It is important to note that ARPE can be used to compare calibration results between data sets with different orders of magnitudes, while RMSE can be used for comparing calibration results within the same or similar data sets and will attach disproportionate weight to the equity tranche spreads. Also recall that the calibration optimises (6.1).
### Parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Jump-Diffusion</th>
<th>Diffusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma$</td>
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<td>0.18</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.11</td>
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<tr>
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<td>$E[Y-1]$</td>
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<td>$\text{Var}[Y-1]$</td>
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<td>$\rho_{X_i</td>
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Table 2: Parameters estimated for 22 February 2007, overall correlation $\rho_{X_i|X_j}$ as in (4.5).

### Measure of Fit

<table>
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<tr>
<th>Measure of Fit</th>
<th>Jump-Diffusion</th>
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<tr>
<td>RMSE</td>
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<tr>
<td>ARPE</td>
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<td>0.42</td>
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</table>

Table 3: Measures of fit for 22 February 2007. Root Mean Square Error (RMSE) is defined by (6.4), Average Relative Percentage Error (ARPE) is given by (6.3).

#### 6.3.2 Diffusion Model

Calibration results for the diffusion model are given in Tables 1, 2, 3, 4, 5 and 6. Root mean square errors (RMSE) are very high, especially for 5 December 2008, but as was noted before, this is partially a result of higher spreads for 5 Dec 2008 compared to 22 Feb 2007.

For 5 Dec 2008, the estimated correlation coefficient, $\rho$, is much higher than obtained for 22 Feb, 2007. High correlation not only increases the probability of companies to default together, but also to survive together. Hence, in order to fit equity tranches, a low correlation is needed, whereas high correlation for the senior tranches. This can be observed in Figure 3 in [5], showing the implied correlation for each tranche. Hence, in a diffusion model, there is usually a trade-off between matching equity and senior tranches. The low correlation coefficient for 22 Feb 2007, results in zero spreads for super senior tranches, (22% - 100%), while equity tranches are underpriced. For 5 Dec 2008, the model produces non-zero spreads for super senior tranches, owing to a higher correlation coefficient, but still both super senior and equity tranches are underpriced.

#### 6.3.3 Jump-Diffusion Model

Tables 1, 2, 3, 4, 5 and 6 show calibration results for the jump-diffusion model. Compared to the diffusion model, fit error measures for both data sets are much lower. What is more, the jump-diffusion model gives spreads of roughly the correct magnitude for all tranches. However, still some tranches are slightly underpriced, whereas other tranches are somewhat overpriced, albeit to a lesser extent than in the diffusion case.

In the jump-diffusion model, apart from $\rho$ and $\sigma$, also $\lambda$, $\mu_Y$ and $\sigma_Y$ affect multiple defaults. Most importantly, unlike $\rho$, higher absolute values of jump parameters, with $\mu_X < 0$, always lead to more expected defaults. Therefore, the trade-off between matching particular tranches is smaller and the overall fit is better in the jump-diffusion model.

Let us now analyse the situation on the market in 22 Feb 2007 and 5 Dec 2008, implied by the model parameters. As mention in 4.1, the initial *distances-to-default,*...
T=5 Years

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<tr>
<td>Index</td>
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<tr>
<td>Tranche 0%-3%</td>
<td>71.5%</td>
<td>76.5%</td>
<td>47%</td>
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<tr>
<td>Tranche 3%-6%</td>
<td>1576.3</td>
<td>1652</td>
<td>1158</td>
</tr>
<tr>
<td>Tranche 6%-9%</td>
<td>811.5</td>
<td>822</td>
<td>825</td>
</tr>
<tr>
<td>Tranche 9%-12%</td>
<td>506.1</td>
<td>504</td>
<td>636</td>
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<tr>
<td>Tranche 12%-22%</td>
<td>180.3</td>
<td>305</td>
<td>415</td>
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<tr>
<td>Tranche 22%-100%</td>
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<td>77</td>
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T=7 Years

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<td>198</td>
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<td>72.9%</td>
<td>78.8%</td>
<td>50%</td>
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<td>Tranche 3%-6%</td>
<td>1473.2</td>
<td>1557</td>
<td>1101</td>
</tr>
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<td>Tranche 6%-9%</td>
<td>804.2</td>
<td>805</td>
<td>805</td>
</tr>
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<td>Tranche 9%-12%</td>
<td>512.4</td>
<td>511</td>
<td>637</td>
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<td>Tranche 12%-22%</td>
<td>182.6</td>
<td>311</td>
<td>434</td>
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<td>Tranche 22%-100%</td>
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<td>74</td>
<td>66</td>
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T=10 Years

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<td>188</td>
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<tr>
<td>Tranche 0%-3%</td>
<td>73.8%</td>
<td>81.2%</td>
<td>51.7%</td>
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<tr>
<td>Tranche 3%-6%</td>
<td>1385.5</td>
<td>1462</td>
<td>1016</td>
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<tr>
<td>Tranche 6%-9%</td>
<td>824.7</td>
<td>768</td>
<td>755</td>
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<tr>
<td>Tranche 9%-12%</td>
<td>526.1</td>
<td>500</td>
<td>607</td>
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<tr>
<td>Tranche 12%-22%</td>
<td>174.1</td>
<td>310</td>
<td>427</td>
</tr>
<tr>
<td>Tranche 22%-100%</td>
<td>76.3</td>
<td>69</td>
<td>70</td>
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Table 4: Calibration results for 5 December 2008, iTraxx Main Series 10 index. Units are basis points (bps) if not stated otherwise. Estimated parameters are given in Table 5, measures of fit in Table 6. We assume that $r = 0.033$, $R = 0.4$.

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<tr>
<td>$\sigma$</td>
<td>0.13</td>
<td>0.18</td>
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<tr>
<td>$\rho$</td>
<td>0.35</td>
<td>0.8</td>
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<td>$\lambda$</td>
<td>0.04</td>
<td>-</td>
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<td>$E[Y_{-1}]$</td>
<td>-0.5</td>
<td>-</td>
</tr>
<tr>
<td>$Var[Y_{-1}]$</td>
<td>0.17</td>
<td>-</td>
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<tr>
<td>$\rho_{\chi^1 \chi^2}$</td>
<td>0.83</td>
<td>-</td>
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Table 5: Parameters estimated for 5 Dec, 2008, overall correlation $\rho_{\chi^1 \chi^2}$ as in (4.5).
obtained from CDS spreads, indicate the current state of the market. As expected, on average, \( x_0 \) is lower for December 2008, around 4.0, whereas for February 2007, it is 4.3. It may seem a bit surprising that the difference is so small, though a better picture is given by the tails of the distribution of initial \textit{distances-to-default}. For the former data set, for 6\% of the companies \( x_0 \) is in (0,1], i.e. 6\% of the companies in the portfolio are already in a bad financial situation, and for 26\% \( x_0 \) is in (0,2], whereas for none of the firms in the latter data set \( x_0 \) is in (0,2]. This well reflects the situation in the market during a crisis and in normal conditions.

The jump intensity, \( \lambda \), is 0.042 for 22 Feb 2007 and 0.035 for 5 Dec 2008 respectively. This implies that under the risk neutral measure a market crash is expected to happen every 25 years for 22 Feb 2007, and every 34 years for 5 Dec 2008. For the former data set, market turbulences will be benign: if a jump occurs, on average a company’s assets value will fall by 7\%. For the latter, market conditions will worsen much more severely: if a jump occurs, a company’s asset value will drop on average by 50\%. Both results agree with the adaptive expectation hypothesis, according to which in a quiet economic conditions market participants anticipate that the good economic climate will be continued, while after a serious market crash, another serious crash is expected.

To the best of our knowledge, a similar calibration exercise is found only in [32] and [19], however the latter study does not discuss the parameters obtained. In [32], a jump-diffusion model with log-normally distributed jumps was calibrated to market data from 8 August 2005. The obtained jump intensity is very small, \( \lambda = 0.0012 \), while the mean jump size is -96\%. Under these settings, on average there is a complete market wipe-out every 870 years. Such a market situation would appear hard to reconcile with the timeframes market participants operate in, or even the age of credit markets and the economy as we know it.

### 6.4 Implied Dependence between Companies

We now study the dependence between companies. The overall correlation, which shows the average tendency of \textit{distances-to-default} to move together, is given in Tables 2 and 5. The first observation is that the correlation is clearly much higher in 2008 than in 2007. For 5 Dec 2008, the overall correlation for the estimated parameters is roughly as big as \( \rho \) for the diffusion model, while for 22 Feb 2007, the total correlation under jump-diffusion is smaller than \( \rho \) in the diffusion model.

However, if a negative jump occurs, dependence between firms, especially the tendency of companies to default together, will increase. This feature of correlation in the jump-diffusion model is captured by a \textit{crisis scenario correlation}, defined in Section 4. Table 7 shows the \textit{crisis scenario correlation} for both data sets. If a negative jump occurs within the first 5 years of the contract, for the 5 Dec 2008 data set, the correlation will be very high, close to 0.9 for all maturities, which means that if a market shock happens, the

<table>
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<tr>
<th>Measure of Fit</th>
<th>Jump-Diffusion</th>
<th>Diffusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMSE</td>
<td>2.42</td>
<td>8.95</td>
</tr>
<tr>
<td>ARPE</td>
<td>0.16</td>
<td>0.35</td>
</tr>
</tbody>
</table>

Table 6: Measures of fit for 5 December 2008. Root Mean Square Error (RMSE) is defined by (6.4), Average Relative Percentage Error (ARPE) is given by (6.3).
tendency of companies to default together will be very strong. Provided that a negative jump occurs, a company’s asset value will drop on average by 57%. Hence, the market expects that if a shock happens, it will trigger many defaults. The scenario has a risk neutral probability of 12.5%.

For the pre-crunch environment, the crisis scenario correlation is low, 0.15, which means that the tendency of companies to default together is slight. On the condition that a negative jump occurs, on average a company’s asset value will drop by 8%, where the risk neutral probability of such scenario is 16.1 %. Hence, it is expected that if a shock happens, its consequences will be benign, and also that such downward fluctuations are quite probable.

These results for the crisis scenario correlation indicate that default correlation should be high for 5 December 2008 and much lower for 22 February 2007. As one can see in Fig. 5, this is indeed the case. We calculated numerically the default correlation both for the jump-diffusion and also for the diffusion model. In order to show the impact of jumps on the default correlation, we also calculate default correlation for a diffusion model with \( \rho \) and \( \sigma \) obtained from a calibration of the jump-diffusion model. Since the probability of a joint default is very small, in particular for small \( t \) and for models calibrated to data from 22 Feb 2007, we used as many as \( 10^9 \) simulations to obtain accurate results.

For both data sets, one can see that jumps strongly affect the default correlation, especially for 5 December 2008, where without jumps, default correlation is smaller than 0.2, whereas with jumps it is as high as around 0.9. For this data set, high default correlation indicates a high probability of multiple defaults. For 22 February 2007, the coefficient is much lower, for \( t \geq 5 \) around 0.05 for all three models. For small \( t \), default correlation equals zero for diffusion models, whereas for jump-diffusion it is close to 0.4. This clearly shows that for small \( t \) default can be triggered only by jumps, and since we assume common jump times, both companies default, hence the high value of the coefficient. As time increases, individual factors start to gain importance and as a result the default correlation decreases.

### Table 7: Crisis scenario correlations, expected falls during crises and risk neutral probabilities

The correlation coefficients, \( \rho_{X_i X_j}^{CS} \), are calculated on the assumption that within the first 5 years a negative jump occurs. The expected fall of asset values is \( E[Y - 1|Y < 1] \), the risk neutral probabilities of market turbulences \( P_{NJ} = P(N_t = 1, \Pi < 0) \).

<table>
<thead>
<tr>
<th>Years</th>
<th>22 February 2007</th>
<th>5 December 2008</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \rho_{X_i X_j}^{CS} )</td>
<td>( P_{NJ} )</td>
</tr>
<tr>
<td>5</td>
<td>0.15</td>
<td>16.1%</td>
</tr>
<tr>
<td>7</td>
<td>0.14</td>
<td>20.7%</td>
</tr>
<tr>
<td>10</td>
<td>0.14</td>
<td>26.1%</td>
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</table>

\[
\mathbb{E} \left[ \frac{dA_t}{A_t} \right]
\]

<table>
<thead>
<tr>
<th>22 February 2007</th>
<th>5 December 2008</th>
</tr>
</thead>
<tbody>
<tr>
<td>-8%</td>
<td>-57%</td>
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</table>
Figure 5: Default correlation for calibrated jump-diffusion and diffusion, and for diffusion with \( \sigma \) and \( \rho \) the same as for jump-diffusion, calculated both for 22 February 2007 and 5 December 2008

7 Conclusions

In this article, we present a multi-name jump-diffusion model, we show how the model can be calibrated to individual CDSs by means of semi-analytical formulae for survival probabilities, derive a numerical method for basket credit derivatives based on a large basket approximation and Monte Carlo finite difference solution of the resulting SPDE, and outline an algorithm for calibration to CDO index and tranche spreads. We provide an economic interpretation of the model and its pricing performance.

The jump-diffusion model proves flexible enough to provide a rough fit of market data in vastly different scenarios, with only a small number of parameters, and shows clear improvements over a pure diffusion model. To employ the model for pricing and risk management in practice, it is necessary to extend the model to allow a richer dependence structure. This can be achieved by modelling the correlation as a function of time and tranche losses, accounting for contagion effects in the markets. Similar to local volatility models, such a local correlation model could be specified to provide a perfect fit to tranche spreads, while the correlation does not impact index and CDS spreads. The jump component will give a realistic term-structure for short maturities. Lastly, Fouque et al. [10] report good results for multi-variate stochastic volatility models, which would also fit naturally in the framework presented here.

References


